Discrete probability, Part 2

Lecture Notes in Math 212 Discrete Mathematics

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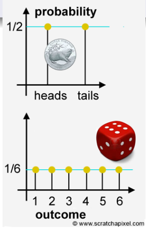
Uniform distribution

So far, we considered the problems in which all outcomes in the sample space S have the same probability $\frac{1}{n}$, where $n = |S|$. In these cases, the probability of an event $E \subset S$ can be found as the sum of probabilities $p(s) = \frac{1}{n}$ of all elements $s \in E$ $\quad p(E) = \sum_{s \in E} p(s) = \frac{|E|}{|S|}$.

Uniform distribution of probabilities in a set S with n elements

assigns probability $\frac{1}{n}$ to each element of S. An experiment of selecting each element (outcome) $s \in S$ with the same probability $\frac{1}{n}$ is called selecting at random.

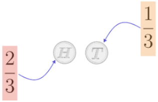


• As we flip a fair coin, outcomes “heads” and “tails” appear with the same probability $\frac{1}{2}$.

• As we roll a fair die, each outcome (any of the numbers $1, 2, \ldots, 6$) appears with probability $\frac{1}{6}$.

General probability distribution

Now we consider a more general situation, in which probabilities of different outcomes are not the same.



Examples

- For a biased (unfair) coin H (heads) may appear two times more often than T (tails). This corresponds to probabilities $p(H) = \frac{2}{3}$, $p(T) = \frac{1}{3}$.
- A biased (loaded) die, with probability distribution $p(4) = p(6) = \frac{1}{4}$ and $p(1) = p(2) = p(3) = p(5) = \frac{1}{8}$, shows 4 and 6 twice more often then the other values.

On any sample space S we consider a function $p : S \to \mathbb{R}$ called probability distribution, which assign to each outcome $s \in S$ its probability $p(s)$.

Probability distribution must satisfy the following two properties

- $0 \leq p(s) \leq 1$ for each $s \in S$ (positivity property),
- $\sum_{s \in S} p(s) = 1$ (normalization: the sum of all probabilities is 1).

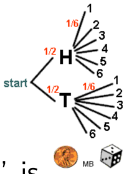
Compound event probability

Compound event

is an event consisting of two or more simple events which happened together. If A and B are simple events, then "A and B" is compound.

Product rule

If two simple events are independent, then the probability of the compound event is the product: $p(A \text{ and } B) = p(A) \cdot p(B)$.



•If we flip a coin and roll dice, then the probability to get "H" (heads) and "6" is $p(H) \cdot p(6) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$.

• If we flip a fair coin 3 times, then the probability to get "HHH" is $p(HHH) = p(H)^3 = \frac{1}{8}$. Similarly, $p(HHT) = \frac{1}{8}$, etc.

• If a coin is biased with $p(H) = \frac{2}{3}$, $p(T) = \frac{1}{3}$, then $p(HHH) = (\frac{2}{3})^3 = \frac{8}{27}$, but $p(HHT) = (\frac{2}{3})^2 \cdot \frac{1}{3} = \frac{4}{27}$, $p(HTT) = \frac{2}{27}$, etc.

If events A and B disjoint (exclusive), then $p(A \text{ and } B) = 0$ (disjoint events cannot happen at the same time).

Examples with unfair coins and dies

Unfair coin

A coin is biased with the probability of heads $\frac{2}{3}$. What is the probability that exactly three heads come up when the coin is flipped five times ?

Solution. *The outcomes with three heads are HHHTT, HHTHT,...,
totally* $\binom{5}{3} = 10$ *possibilities (from 5 flips we choose 3 which give heads). The probability of each of these 10 outcomes is* $(\frac{2}{3})^3 \cdot (\frac{1}{3})^2 = \frac{8}{241}$, *so, totally we obtain probability* $10 \cdot \frac{8}{241} = \frac{80}{241}$

Unfair die

A die is unfair: 6 appears twice more often than any other number. What is the probability to obtain in sum 10 after rolling this die twice ?

Solution. *Let us denote by p the probability* $p(1) = p(2) = \cdots = p(5)$. *Then* $p(6) = 2p$ *and since* $p(1) + \cdots + p(6) = 5p + 2p = 1$ *we obtain* $p = \frac{1}{7}$. *The sum* 10 *can be obtained as* $6 + 4$, $5 + 5$, *and* $4 + 6$, *with the corresponding probabilities* $(2p) \cdot p$, $p \cdot p$, *and* $p \cdot (2p)$. *Totally, we obtain probability* $2p^2 + p^2 + 2p^2 = 5p^2 = 5(\frac{1}{7})^2 = \frac{5}{49}$.

Picking marbles in a bowl

Consecutive and alternative choice of marbles

- A bowl contains 12 red marbles, 5 blue marbles and 13 yellow marbles. Find the probability of drawing a blue marble and then a yellow one.
  Solution. *The probability to get a blue marble after the first drawing is* $\frac{5}{30} = \frac{1}{6}$, *because there are totally* 30 *marbles. Since after the first drawing 29 marble remains, the probability to get a yellow marble after the second drawing is* $\frac{13}{29}$. *So, the answer is* $\frac{1}{6} \cdot \frac{13}{29} = \frac{13}{174}$.

- One bowl contains 4 red marbles and 3 blue ones. Another bowl contains two red and one blue marbles. One chooses randomly any of the two bawl and picks a marble in it. Find the probability that this marble is red.
  Solution. *By the first choice, one of two bowls is chosen: probability to choose each one is* $\frac{1}{2}$. *By the second choice a marble is chosen. It will be red with probability* $\frac{4}{7}$ *in the first case and* $\frac{2}{3}$ *in the second. Hence, the overall probability to choose a red marble is* $\frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{7} + \frac{1}{3} = \frac{13}{21}$.