Math 112 Discrete Mathematics Lecture Notes Part I

CONSTANT COEFFICIENT LINEAR HOMOGENEOUS RECURRENCE RELATIONS

RECURSIVE RELATION

There are various ways of describing a sequence u_0, u_1, u_2,... of real numbers. An obvious way is to express general term a_n term as an explicit function of n, that is $u_n=f(n)$. For example, when we write

$$a_n=n^2$$

one can compute any term of the sequence directly, such as

$$a_0=0,\ a_1=1,\ a_{15}=225,\ a_{312}=970944,\dots .$$

An alternative way to define the n-th term is to write a relation which gives n-th term by means of the preceding terms. For example, consider the sequence $\{b_n\}$

$$2 \quad 5 \quad 3.5 \quad 4.25 \quad 3.875, \dots$$

where each term, starting with b_2, is equal to the arithmetic mean of the two preceding terms, that is

$$b_n = (b_{n-1}+b_{n-2})/2.$$

Such a sequence is said to be defined recursively (or inductively) and the expression $u_n=F(u_{n-1},u_{n-2},\dots,u_{n-t})$ which relates u_n to the preceding terms is called **a recursive relation** (or a **recursion**).

INITIAL TERMS

Consider the sequence $\{b_n\}$ defined above with the recursion $b_n=(b_{n-1}+b_{n-2})/2$. Notice that, in order to compute b_{10}, we have to first compute b_9 and b_8. Then we have to compute b_7 and b_6 ... . This means that, when a sequence is defined recursively, in order to run necessary computations, we must be given a number of initial terms explicitly to start with. For the sequence above if b_0 and b_1 are given, then the rest of sequence can be computed using the recurrence.

Explicitly given first few such terms of the sequence are called the **initial terms** of the sequence.

It follows that the proper way of defining a sequence recursively is to give a recursive relation provided with sufficiently many initial terms. Thus, the expression

$$b_0=2,\ b_1=5 \text{ and } b_n=(b_{n-1}+b_{n-2})/2 \text{ for } n>1$$

defines the sequence $\{b_n\}$ without any ambiguity.

Let $\{c_n\}$ be the sequence defined by setting $c_0=1$ and $c_n=c_0+\ldots+c_{n-1}$ for $n>0$. Then first few terms of $\{c_n\}$ are

$$c_0 = 1$$
$$c_1 = c_0 = 1$$
$$c_2 = c_0+c_1 = 2$$
$$c_3 = c_0+c_1+c_2 = 4$$
$$c_4 = c_0+c_1+c_2+c_3 = 8$$
$$\ldots$$

ORDER OF A RECURRENCE RELATION

Returning back to above examples, in the recursion given for $\{b_n\}$ the farthest term related with b_n is b_{n-2} and difference of indices of these terms is $n-(n-2)=2$. On the other hand, in the recursion for $\{c_n\}$, the farthest term related with c_n is c_0 and difference of indices is $n-0=n$.

When a sequence $\{u_n\}$ is defined with a recursive relation, if the difference of n and the index of the farthest term related with u_n is constant (as in the case of $\{a_n\}$), this constant value is called the ***order*** (or ***degree***) of the recursion. When the order is defined, the number of initial terms required to construct the sequence is equal to the order of recursion.

Order of the recursion given for $\{b_n\}$ is 2, whereas order of the recursion for $\{c_n\}$ is undefined.

Example 1. Let $\{u_n\}$ be defined with $u_n=u_{n-1}+u_{n-2}+\ldots+u_{[n/2]}$, $u_0=1$. Then

$u_1=u_0=1$

$u_2=u_1=1$

$u_3=u_2+u_1=2$

$u_4=u_3+u_2=3$

$u_5=u_4+u_3+u_2=6$

...

and order of the recursion is undefined.

Example 2. Order of the recursion $v_n=v_{n-1}v_{n-2}-v_{n-5}$ is 5 and to define the sequence $\{v_n\}$ uniquely, 5 initial terms should be given. Say $v_0=30$, $v_1=20$, $v_2=10$, $v_3=6$ and $v_4=5$ then

$v_5 = v_4v_3-v_0 = 0$

$v_6 = v_5v_4-v_1 = -20$

$v_7 = v_6v_5-v_2 = -10$

$v_8 = v_7v_6-v_3 = 194$

$v_9 = v_8v_7-v_4 = -1945$

...

SOLUTION OF A RECURRENCE RELATION

Any sequence whose terms (except the initial terms) satisfy a recurrence relation is called a **(particular) solution** of that recurrence relation.

To solve a recurrence relation, subject to sufficiently many initial terms, means to find an explicit expression for general term of the sequence whose terms (except the initial ones) satisfy the recurrence.

Example 3. Consider the recurrence relation $u_n = nu_{n-1}$. For the initial term $u_0=3$, we compute first few terms of the sequence:

$u_1=1u_0 = 3 \quad = 3\text{x}1$

$u_2=2u_1 = 6 \quad = 3\text{x}2$

$u_3=3u_2 = 18 = 3\text{x}6$

$u_4=4u_3 = 72 = 3\text{x}24$

$u_5=5u_4 = 360= 3\text{x}120$

...

One can easily suggest that the general term is

$$u_n=3\cdot n!$$

Of course this suggestion needs to be proved. Indeed, by mathematical induction one can easily prove the claim and we leave this part as an exercise.

Example 4. Consider the recurrence $a_n = na_{n-1} + (n-1)!$ and the initial condition $a_1=1$. Then we have

$u_1 = 1$

$u_2 = 2\cdot 1 + 1!$

$u_3 = 3! + 3\cdot 1! + 2!$

$u_4 = 4! + 4\cdot 3\cdot 1 + 4\cdot 2\cdot 1 + 3!$

$u_5 = 5! + 5\cdot 4\cdot 3\cdot 1 + 5\cdot 4\cdot 2\cdot 1 + 5\cdot 4\cdot 3\cdot 1 + 4!$

$u_6 = 6! + 6\cdot 5\cdot 4\cdot 3\cdot 1 + 6\cdot 5\cdot 4\cdot 2\cdot 1 + 6\cdot 5\cdot 4\cdot 3\cdot 1 + 6\cdot 4\cdot 3\cdot 2\cdot 1 + 5!$

We notice that u_6

$$u_6 = \frac{6!}{1} + \frac{6!}{2} + \frac{6!}{3} + \frac{6!}{4} + \frac{6!}{5} + \frac{6!}{6}$$

or

$$u_6 = 6!\,H_6$$

where $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$. Then we may claim that

$$u_n = n!H_n.$$

Example 5. Find the solution of $u_n = (n-1)(\underline{u}_{n-1} + u_{n-2})$ subject to the initial conditions $a_0 = a_1 = 1$. To have an insight about the sequence we compute a few terms:

$u_0 = 1$

$u_1 = 1$

$u_2 = 1(1 + 1) = 2$

$u_3 = 2(2 + 1) = 6$

$u_4 = 3(6 + 2) = 24$

$u_5 = 4(24 + 6) = 120$

$u_6 = 5(120 + 24) = 720$

We can suggest that $u_n = n!$

Now we prove by induction that indeed $u_n = n!$

Basic step is obvious since

$u_0 = 1 = 0!$

$u_1 = 1 = 1!$

Now assume that the equality $u_n = n!$ holds for all integers n with $1 < n < k$ for some fixed integer $k > 1$. Then by recursion we have

$$u_k = (k-1)(\underline{u}_{k-1} + u_{k-2})$$

Assumption of induction implies that $u_{k-1} = (k - 1)!$ and $u_{k-2} = (k - 2)!$ so

$$\begin{aligned} u_k &= (k-1)[(k - 1)! + (k - 2)!] \\ &= (k-1)[\,(k - 2)!(k - 1+1)] \\ &= (k-1)(k-2)!k \\ &= k! \end{aligned}$$

Consequently, our suggestion is correct.

We have seen a recurrence relation given together with sufficiently many initial conditions determines a sequence uniquely. By changing initial terms, we can obtain different sequences all of whose terms satisfy the given recurrence. For example, each of the sequences below are constructed according to the relation $u_n = (u_{n-1}+u_{n-2})/2$:

sequence	initial terms	first 6 terms of the sequence
$\{b_n\}$	$b_0=2,\quad b_1=5$	2 5 3.5 4.25 3.875 4.0625 ...
$\{b'_n\}$	$b'_0=5,\quad b'_1=2$	5 2 3.5 2.75 3.125 2.9375 ...
$\{b''_n\}$	$b''_0=0,\quad b''_1=4$	0 16 8 12 10 11 ...
$\{b'''_n\}$	$b'''_0=3,\quad b'''_1=3$	3 3 3 3 3 3 ...

If we have given only a recurrence relation, not the initial terms, then it is possible to find infinitely many sequences (depending on choice of the initial terms), whose terms (starting with a specific index) satisfy the given recurrence.

Example 6. Consider the recurrence relation of Example 3 without any initial terms provided. Then we may let u_0=A. Then

$$\begin{aligned} u_1 &= 1u_0 = A \\ u_2 &= 2u_1 = 2A \\ u_3 &= 3u_2 = 6A \\ u_4 &= 4u_3 = 24A \\ u_5 &= 5u_4 = 120A \\ &\dots \end{aligned}$$

It is seen that we can claim that the general term is

$$u_n = A(n+1)!$$

Claim can be proved to be true by mathematical induction.

The 'A' appearing in the solution is the parameter which is related with the initial term, in fact in this example $A = a_0$.

Example 7. Consider the recurrence relation $u_n = 2u_{n-1} - u_{n-2}$. Depending on the initial terms u_0 and u_1 we compute the remaining terms as

$$u_2=2u_1 - u_0 = 2u_1 - u_0$$
$$u_3=2u_2 - u_1= 2(2u_1 - u_0) - u_1 = 3u_1-2u_0$$
$$u_4=2u_3 - u_2 = 2(3u_1 - 2u_0) - (2u_1 - u_0) = 4u_1 - 3u_0$$
$$u_5=2u_4 - u_3 = 2(4u_1 - 3u_0) - (3u_1 - 2u_0) = 5u_1 - 4u_0$$
$$\ldots$$

We claim that the general term is

$$u_n=nu_1-(n-1)u_0$$

which can be rearranged as $u_n=n(u_1- u_0)+ u_0$. If we put $A = u_1- u_0$ and we see that any (particular) solution of the recurrence $u_n = 2u_{n-1} - u_{n-2}$ is of the form

$$u_n=A_n+B.$$

In the last example the expression $u_n=A_n+B$ has the property that for any choice of parameters A and B, it gives general term of a particular solution (sequence) of the recursion $u_n = 2u_{n-1} - u_{n-2}$. Such an expression is called the general solution of the given recurrence relation.

To solve a given recurrence relation means to find the general solution, that is, an expression which depends on a certain number of parameters such that every particular solution of the recurrence relation can be obtained by assigning proper values to these parameters.

In general, to solve a recurrence relation is a difficult problem and we do not have general methods. In this course we will be interested in only a specific class of recurrence relations, so called ‘constant coefficient linear relations’. You will be responsible of finding solutions of only this class.

End of part I