Jordan KdV Systems and Painlevé Property

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Received August 18, 1996

The Painlevé property of Jordan KdV systems in two dimensions is studied. It is shown that a subclass of these equations on a nonassociative algebra possesses the Painlevé property.

1. INTRODUCTION

Svinolupov (1991) introduced many-field Korteweg–de Vries (KdV) equations

\[ u_i^{\prime} = u_{xxx} + a_{jk} u_j u_k, \quad i = 1, \ldots, N \]  \hspace{1cm} (1.1)

where \( u_i \) depend on variables \( x \) and \( t \), and \( a_{jk} \) is a set of constants symmetric with respect to the subscripts. He showed that there is a one-to-one correspondence between such equations and Jordan algebras. Specifically, Jordan KdV systems have an infinite algebra of generalized symmetries, an infinite series of local conservation laws, and a recursion operator. The systems corresponding to simple Jordan algebras are called irreducible. In two dimensions all the systems related to two-dimensional Jordan algebras contain a scalar KdV equation and a linear equation which are not really coupled (Svinolupov, 1991, 1994).

In this work we consider the system (1.1) for \( N = 2 \). We apply the Painlevé test for partial differential equations introduced by Weiss et al. (1983) to the system of coupled KdV equations without \textit{a priori} assumptions about the algebraic nature of the system. We find the sets of constants \( a_{jk} \) for which the system (1.1) possesses the Painlevé property. A subclass of these equations on a nonassociative algebra has the Painlevé property. By

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using the truncated expansions of the solutions we also obtain the auto-Bäcklund transformations for these equations.

2. SYMMETRY APPROACH

The recursion operator for the scalar Korteweg–de Vries equation ($N = 1$) is given in Olver (1993) as

\[
L = D^2 + \frac{2}{3}u + \frac{1}{3}u_x D^{-1}
\]  

(2.1)

where $D \equiv d/dx$. An analogous operator also exists for Jordan systems. In Svinolupov (1991) it is stated as a theorem that any Jordan system (1.1) is integrable and possesses a formal recursion operator:

\[
L = D^2 + \frac{2}{3}a(i)u^i + \frac{1}{3}u_x D^{-1}a(i)
+ \frac{1}{6}u^kD^{-1}u^lD^{-1}[a^a_k a(n) - a(k)a(j)]
\]  

(2.2)

where the matrices $a(j)$ are determined by the formula $(a(j))_k^i = a_{jk}^i$ and the constants $a_{jk}$ are the structure constants of a Jordan algebra satisfying the identities

\[
a^a_{jk}(a^i_{nr} a^r_{ns} - a^i_{mr} a^r_{ns}) + a^a_{kn}(a^i_{nr} a^j_{ns} - a^j_{mr} a^i_{ns})
+ a^a_{mj}(a^i_{nr} a^k_{ns} - a^k_{mr} a^i_{ns}) = 0
\]  

(2.3)

We consider a system of two nonlinear equations of the form

\[
u_t = u_{xxx} + c_1 u u_x + c_2 (uv_x + vu_x) + c_3 v v_x
\]

\[
u_t = v_{xxx} + d_1 u u_x + d_2 (uv_x + vu_x) + d_3 v v_x
\]  

(2.4)

where

\[
c_1 = a_{11}, \quad d_1 = a_{11}, \quad u = u^1
\]

\[
c_2 = a_{21} = a_{12}, \quad d_2 = a_{12} = a_{21}, \quad v = u^2
\]  

(2.5)

The recursion operator (2.2) for this problem can be expressed as a $2 \times 2$ matrix whose components are

\[(L)_{11} = D^2 + \frac{2}{3}(c_1 u + c_2 v) + \frac{1}{3}(u_x D^{-1}c_1 + v_x D^{-1}c_2)
- \frac{1}{6}(u D^{-1})(v D^{-1})F_1 - \frac{1}{6}(v D^{-1})^2 F_2
\]  

(2.6)

\[(L)_{12} = \frac{2}{3}(c_2 u + c_3 v) + \frac{1}{3}(u_x D^{-1}c_2 + v_x D^{-1}c_3)
+ \frac{1}{6}(u D^{-1})^2 F_1 + \frac{1}{6}(v D^{-1})(u D^{-1})F_2
\]  

(2.7)
\[ (L)_{21} = \frac{2}{3}(d_1 u + d_2 v) + \frac{1}{3}(u_x D^{-1} d_1 + v_x D^{-1} d_2) \]
\[ - \frac{1}{9}(u D^{-1}) (v D^{-1}) F_3 + \frac{1}{9}(v D^{-1})^2 F_1 \] (2.8)

\[ (L)_{22} = D^2 + \frac{2}{3}(d_1 u + d_2 v) + \frac{1}{3}(u_x D^{-1} d_2 + v_x D^{-1} d_3) \]
\[ + \frac{1}{9}(u D^{-1})^2 F_3 - \frac{1}{9}(v D^{-1}) (u D^{-1}) F_1 \] (2.9)

where

\[ F_1 = c_3 d_1 - c_2 d_2, \quad F_2 = c_2^2 - c_1 c_3 + c_3 d_2 - c_2 d_3 \] (2.10)

\[ F_3 = d_1 d_3 - d_2^2 + c_1 d_2 - c_2 d_1 \]

and the structure constants \( c_i \) and \( d_i \) satisfy the following identities:

\[ (c_1 - 2d_2) F_1 = 0, \quad (c_1 - 2d_2) F_2 = 0, \quad (c_1 - 2d_2) F_3 = 0 \] (2.11)

\[ (d_3 - 2c_2) F_1 = 0, \quad (d_3 - 2c_2) F_2 = 0, \quad (d_3 - 2c_2) F_3 = 0 \]

\[ d_1 F_1 = 0, \quad d_1 F_2 = 0, \quad d_1 F_3 = 0 \]

\[ c_3 F_1 = 0, \quad c_3 F_2 = 0, \quad c_3 F_3 = 0 \]

If \( F_1, F_2, F_3 \) vanish, the recursion operator reduces to a form similar to (2.1). This case corresponds to an associative algebra in which the system (2.4) decouples.

### 3. Painlevé Analysis

A partial differential equation has the Painlevé property when its solutions are single-valued about the movable singularity manifold. If the singularity manifold is determined by

\[ \phi(\phi^0, x^1, \ldots, x^n) = 0 \] (3.1)

and \( u^a (a = 1, \ldots, N) \) satisfy a system of partial differential equations \((N\text{-equations})\), then the Painlevé expansion is given by

\[ u^a = \phi^{\alpha_a} \sum_{k=0}^{\infty} u_k^a (\phi^0, x^1, \ldots, x^n) \phi^k \] (3.2)

where \( u_k^a \) are analytic functions of \((\phi^0, x^1, \ldots, x^n)\) in a neighborhood of the manifold (3.1). The substitution of (3.2) into the partial differential equations under consideration determines the possible values of \( \alpha_a \) and gives the recursion relations for \( u_k^a \). A set of partial differential equations is said to have the Painlevé property in the sense of Weiss et al. provided \( \alpha_a \) are integers, the recursion relation are consistent, and the series expansion (3.2)
contains the correct number of arbitrary functions. Applying the Painlevé analysis to equations (2.4), we obtain the following:

(i) Leading order analysis. Substituting $u = u_0\phi^{\alpha_1}$ and $v = v_0\phi^{\alpha_2}$ into the leading terms of (2.4), we have $\alpha_1 = \alpha_2 = -2$ and the equations for $u_0$ and $v_0$,

\begin{align*}
c_1u_0^2 + 2c_2u_0v_0 + c_3v_0^2 + 12u_0\phi_x^2 &= 0 \quad (3.3) \\
d_1u_0^2 + 2d_2u_0v_0 + d_3v_0^2 + 12v_0\phi_x^2 &= 0 \quad (3.4)
\end{align*}

(ii) Resonances. Substituting

\begin{equation*}
u = u_0\phi^{-2} + \beta_1\phi^{-2}, \quad v = v_0\phi^{-2} + \beta_2\phi^{-2}
\end{equation*}

into the leading terms of equations (2.4) and requiring that $\beta_1$ and $\beta_2$ be arbitrary, we have

\begin{align*}
\{ &\phi_x^4(r - 2)^2(r - 3)^2 + \phi_x^2(r - 2)(r - 3)[u_0(d_2 + c_1) + v_0(d_3 + c_2)] \\
&+ [(u_0c_1 + v_0c_2)(u_0d_2 + v_0d_3) - (u_0c_2 + v_0c_3)(u_0d_1 + v_0d_2)] \} \\
&\times (r - 4)^2 = 0 \quad (3.6)
\end{align*}

The roots of this equation determine the resonances. $r = 4$ is a double root which satisfies the equation identically. We must always have the root $r = -1$, since it represents the arbitrariness of the singularity manifold $\phi(x, t) = 0$. This is possible if

\begin{align*}
144\phi_x^4 + 12\phi_x^2[u_0c_1 + v_0(d_3 + 2c_2)] \\
+ v_0[u_0(d_3c_1 - d_1c_3) + 2v_0(d_3c_2 - d_2c_3)] &= 0 \quad (3.7)
\end{align*}

If this equation is satisfied, we have another root, $r = 6$. Using equations (3.3), (3.4), and (3.7), we find that equation (3.6) becomes

\begin{align*}
12\phi_x^4(r^2 - 5r + 6) + 12\phi_x^2(u_0d_2 - v_0c_2) \\
+ v_0[u_0(d_1c_3 - d_3c_1) + 2v_0(d_2c_3 - d_3c_2)] &= 0 \quad (3.8)
\end{align*}

The roots of this equation are

\begin{align*}
r_1 &= \frac{15\phi_x^2 - \sqrt{6\phi_x^2}}{6\phi_x^2}, \quad r_2 = \frac{15\phi_x^2 + \sqrt{6\phi_x^2}}{6\phi_x^2} \quad (3.9)
\end{align*}

$r_1$ and $r_2$ must be integers, say $n_1$ and $n_2$; then we have the following values of resonances:

\begin{align*}
r &= -1, 4, 4, 6, n_1, n_2 \quad \text{where} \quad n_1 + n_2 = 5 \quad (3.10)
\end{align*}

Now let us examine the different cases for $n_1, n_2$. 
Case 1. Let $n_1 = 0$, $n_2 = 5$; then from equation (3.8) we have

$$72\phi_x^4 + 12\phi_x^2(u_0d_2 - v_0c_2) + v_0[u_0(d_1c_3 - d_2c_1) + 2v_0(d_2c_3 - d_3c_2)] = 0$$  \hspace{1cm} (3.11)

Equations (3.3), (3.4), (3.7), and (3.11) must be solved for $u_0$ and $v_0$. Since one of the roots is zero, the function $u_0$ or $v_0$ must be arbitrary. If $u_0$ is arbitrary, the equations under consideration imply $v_0 = \alpha\phi_x^2 + \beta$, where $\alpha$ and $\beta$ are constants. Requiring that the equations for $u_0$ and $v_0$ be satisfied, we have the following solution:

$$\alpha = -12/d_3, \quad \beta = -u_0d_2/c_2$$

$$d_1 = 0, \quad d_2 = c_1/2, \quad d_3 = 2c_2, \quad c_3 = 0$$

$$u_0 \text{ is arbitrary}, \quad v_0 = \frac{1}{2c_2}(-12\phi_x^2 - u_0c_1)$$  \hspace{1cm} (3.12)

(iii) Arbitrary functions. To discuss the arbitrariness of the functions corresponding to resonance values $-1, 0, 4, 4, 5, 6$, we have to substitute

$$U = \sum_{j=0}^{6} u_j\phi_x^{j-2}, \quad V = \sum_{j=0}^{6} v_j\phi_x^{j-2}$$

into equations (2.4) and obtain the recursion relations for $u_j$ and $v_j$. Solving these relations, we have

$$j = 0 \quad u_0 = \text{arbitrary}; \quad v_0 = -(12\phi_x^2 + c_1u_0)/2c_2$$  \hspace{1cm} (3.13)

$$j = 1 \quad v_1 = \left(\phi_{xx}(12\phi_x^2 - u_0c_1) + \phi_xu_0c_1\right)/2\phi_x^2c_2$$

$$u_1 = \left(\phi_{xx}u_0 - \phi_xu_0c_1\right)/\phi_x^2$$  \hspace{1cm} (3.14)

$$j = 2 \quad v_2 = \left[8\phi_{xxx}\phi_x(-6\phi_x^2 + u_0c_1) + \phi_x^2(36\phi_x^2 - 21u_0c_1) + 18\phi_{xx}\phi_xu_0c_1 + 6\phi_x^2(2\phi_x^2u_0c_1 - u_0c_1) + \phi_x\phi_uu_0c_1\right]/24\phi_x^2c_2$$

$$u_2 = \left[-8\phi_{xxx}\phi_xu_0 + \phi_x(21\phi_xu_0 - 18\phi_xu_0c_1) + \phi_x\phi_uu_0c_1\right]/12\phi_x^2$$  \hspace{1cm} (3.15)

$$j = 3 \quad v_3 = \left[3\phi_{xxt}\phi_x(-4\phi_x^2 + u_0c_1) + 3\phi_{xxxxt}\phi_x(4\phi_x^2 - u_0c_1) + 4\phi_{xxxt}\phi_x(-12\phi_x^2 + 7u_0c_1) - 10\phi_{xxxxt}\phi_xu_0c_1 + \phi_{xx}(36\phi_x^2\phi_x^2 - 39\phi_x^2u_0c_1 + 33\phi_{xxt}\phi_xu_0c_1 + 12\phi_x^4\phi_x - 12\phi_x^2u_0c_1 - \phi_x\phi_uu_0c_1) + \phi_x^2(2\phi_xu_0c_1 - 2\phi_xu_0c_1 + \phi_x\phi_uu_0c_1)/24\phi_x^2c_2ight]$$
\[ u_3 = \left\{-3\phi_x\phi_x^2 u_0 + 3\phi_{xxx}\phi_x^2 u_0 + \phi_{xxx}\phi_x(-28\phi_x u_0 + 10\phi_x u_0) \right. \\
\left. + \phi_x(39\phi_x^2 u_0 - 33\phi_x\phi_x u_0 + 12\phi_x^2 u_{xxx} + \phi_x \phi_x u_0) \right. \\
\left. + \phi_x^2(-2\phi_x u_{xxx} + 2\phi_x u_0 - \phi_x u_{xxx})\right\}/12\phi_x^6 \]  
\[ (3.16) \]

\( j = 4 \) the compatibility conditions are satisfied identically, which means \( u_4 \) and \( v_4 \) are arbitrary functions

\( j = 5 \) \( v_5 = \text{arbitrary} \)

\[ u_5 = \left\{-v_5\phi_x(12\phi_x^2 + u_0 c_1 \phi_x + 4v_0 c_2) - v_4(12\phi_x^2 + u_0 c_1 + 4v_0 c_2) \right. \\
\left. - v_4(12\phi_x\phi_x + u_1 c_1 \phi_x + 4v_1 c_2 \phi_x + u_0 c_1 + 4v_0 c_2) \right. \\
\left. - v_0 c_1 u_4 - u_4(\phi_v v_1 c_1 + v_0 c_1) - 6\phi_v v_3 c_2 \right. \\
\left. - v_3(6\phi_{xxx} + u_1 c_1 + 4v_1 c_2) - v_3(2\phi_{xxx} + 2\phi_x v_2 c_2 \right. \\
\left. + \phi_x u_2 c_1 - 2\phi_x + u_1 c_1 + 4v_1 c_2) - u_3 v_1 c_1 \right. \\
\left. - u_3(\phi_x v_2 c_1 + v_1 c_1 \right. \\
\left. - 2v_{xxx} - v_2(u_2 c_1 + 4v_2 c_2) + 2v_{2x} - v_2 c_1 u_{2x}/\phi_x v_0 c_1 \right\} \]  
\[ (3.17) \]

\( j = 6 \) \( u_6 = \text{arbitrary} \)

\[ v_6 = \left\{-2\phi_x v_0 c_1 u_6 - v_5(36\phi_x \phi_x + 2\phi_x u_1 c_1 + 8\phi_x v_1 c_2 + u_0 c_1 + 4v_0 c_2) \right. \\
\left. - u_5 v_0 c_1 - u_5(2\phi_x v_1 c_1 + v_0 c_1) - 12\phi_x v_{xxx} \right. \\
\left. - v_4(12\phi_x + u_1 c_1 + 4v_1 c_2) \right. \\
\left. - v_4(4\phi_{xxx} + 2\phi_x u_2 c_1 + 8\phi_x v_2 c_2 - 4\phi_x + u_1 c_1 + 4v_1 c_2) \right. \\
\left. - u_4(\phi_x v_1 c_1 + 2\phi_x v_2 c_1) - 2v_{xxx} \right. \\
\left. - v_3(u_2 c_1 + 4v_2 c_2) + 2v_{3x} - u_3 v_2 c_1 - u_2 v_3 c_1 \right. \\
\left. - 2v_2(u_3 c_1 + 4v_3 c_2) - 2\phi_x v_3(u_3 c_1 \right. \\
\left. + 2v_3 c_2)/[2\phi_x(24\phi_x^2 + u_0 c_1 + 4v_0 c_2)] \right\} \]  
\[ (3.18) \]

Since \( \phi, u_0, u_4, v_4, v_5, u_6 \) are arbitrary functions corresponding to the resonances \((-1, 0, 4, 4, 5, 6)\), the system of equations (2.4) with \( c_3 = d_1 = 0, d_3 = 2c_2, d_2 = c_1/2 \) passes the Painlevé test. It is easy to check that the Jordan algebra is nonassociative with these values of \( c_i \) and \( d_i \). In Weiss (1983, 1986) it was shown that the Bäcklund transformations can be obtained
by truncating the expansions (3.2) at constant level terms, that is, $u_j = 0$ if $j \geq 3$, $v_j = 0$ if $j \geq 3$. This is possible if

$$
\begin{align*}
  u_{2t} &= u_{2xxx} + c_1 u_2 u_{2x} + c_2 (u_2 v_{2x} + v_2 u_{2x}) \\
  v_{2t} &= v_{2xxx} + \frac{c_1}{2} (u_2 v_{2x} + v_2 u_{2x}) + 2c_2 v_2 v_{2x}
\end{align*}
$$

(3.19)

Equations (3.13)–(3.15) and (3.19) will be consistent if

$$
\phi_t = \phi_{xxx} - \frac{3}{2} \frac{\phi_{xx}^2}{\phi_x} + \lambda \phi_x, \quad \lambda = \text{const}
$$

(3.20)

which can be formulated in terms of the Schwarzian derivative

$$
\frac{\phi_t}{\phi_x} - \{\phi; x\} = \lambda
$$

(3.21)

and the function $u_0$ must be a solution of the linear equation

$$
\begin{align*}
  2\phi_x^3 u_0 t - 2f_x^3 u_{0xxx} + 12f_{xx}\phi_x^2 u_{0xx} \\
  - \phi_x (18\phi_x^2 + 10\lambda \phi_x^2 - 9\phi_x \phi_t) u_0 \\
  + 2\phi_{xx} (3\phi_x^2 + 8\lambda \phi_x^2 - 9\phi_x \phi_t) u_0 &= 0
\end{align*}
$$

(3.22)

Then,

$$
\begin{align*}
  \phi_t &= u_2 - \frac{1}{\phi} \left( \frac{u_0}{\phi_x} + \frac{u_0}{\phi} \right) \\
  v_t &= v_2 + \frac{\phi_t}{c_2} (\ln \phi)_xx + \frac{c_1}{2c_2 \phi} \left[ \left( \frac{u_0}{\phi_x} \right)_x - \frac{u_0}{\phi} \right]
\end{align*}
$$

(3.23)

will define the Bäcklund transformations for the Jordan KdV system, which generate nontrivial solutions from trivial ones. Note that a particular solution of (3.22) is $u_0 = C\phi_x^2$, where $C$ is a constant.

**Case 2.** If $n_1 = 1$, $n_2 = 4$, the test fails, since the number of arbitrary functions is less than the number of resonances ($-1, 4, 4, 6, 1, 4$).

**Case 3.** Let $n_1 = 2$, $n_2 = 3$; then from equation (3.8) we have

$$
12\phi_x^2 (u_0 d_2 - v_0 c_2) + v_0 [u_0 (d_1 c_3 - d_3 c_1) + 2v_0 (d_2 c_3 - d_3 c_2)] = 0
$$

(3.24)

To solve equations (3.3), (3.4), (3.7), and (3.24) for $u_0$ and $v_0$, let

$$
\begin{align*}
  v_0 &= \alpha \phi_x^2 + \beta \\
  u_0 &= \delta \phi_x^2 + \gamma
\end{align*}
$$
where $\alpha$, $\beta$, $\gamma$, $\delta$ are constants. Substituting these into equations (3.3), (3.4), (3.7), and (3.24), we have

$$\beta = 0, \quad \gamma = 0, \quad \delta \neq 0$$

$$d_1 = -\frac{1}{\delta^2} (2d_2\delta\alpha + d_3\alpha^2 + 12\alpha)$$

$$d_2 = -\frac{1}{\delta^2} (d_3\delta\alpha - c_2\delta\alpha - c_3\alpha^2)$$

$$c_1 = -\frac{1}{\delta^2} (2c_2\delta\alpha + c_3\alpha^2 + 12\delta)$$

$$u_0 = \delta \phi_x^2, \quad v_0 = \alpha \phi_x^2$$  \hspace{1cm} (3.25)

(iiib) Arbitrary functions. Substituting

$$u = \sum_{j=0}^{6} u_j \phi^{j-2}, \quad v = \sum_{j=0}^{6} v_j \phi^{j-2}$$

into equations (2.4), we observe that these equations pass the Painlevé test if

$$d_3 = \frac{1}{\delta(c_2\delta + c_3\alpha)} (c_2^2\delta^2 + 3c_2c_3\delta\alpha + 2c_3^2\alpha^2 + 12c_3\delta)$$  \hspace{1cm} (3.26)

where

$$u_0 = \delta \phi_x^2, \quad v_0 = \alpha \phi_x^2$$

$$u_1 = -\delta \phi_{xx}, \quad v_1 = -\alpha \phi_{xx}$$

$$v_2 = [u_2[\phi_x^2(c_2\delta\alpha + c_3\alpha^2 + 12\delta)]$$

$$+ \delta^2[\phi_x(-4\phi_{xxx} + \phi_x) + 3\phi_{xx}^2]/[\phi_x^2\delta(c_2\delta + c_3\alpha)]$$

$$v_3 = [u_3[\phi_x^4(c_2\delta\alpha + c_3\alpha^2 + 12\delta)]$$

$$+ \delta^2[\phi_x(-2\phi_{xx} + \phi_{xxx})$$

$$+ \phi_x\phi_{xx}(-4\phi_{xxx} + \phi_x) + 3\phi_{xx}^3]/\phi_x^4\delta(c_2\delta + c_3\alpha)$$  \hspace{1cm} (3.27)

The expressions for $u_5$, $v_5$, and $v_6$ are very extensive, therefore are not presented here. The functions $u_2$, $u_3$, $u_4$, $v_4$ are arbitrary, and $u_6$ is also arbitrary if (3.26) is valid. But in this case $F_1$, $F_2$, $F_3$ in (2.11) vanish, where $d_1$, $d_2$, $c_1$ are given in (3.25), which implies that we have an associative algebra. Thus, in this case the system of equations (2.4) decouples.
4. CONCLUSION

We conclude that the system of equations
\[ u_t = u_{xxx} + c_1uu_x + c_2(uvv_x + vuv_x) \]
\[ v_t = v_{xxx} + \frac{c_1}{2}(uv_x + vux) + 2c_2vv_x \]
possesses the Painlevé property, has Bäcklund transformations, and corresponds to a nonassociative Jordan algebra. However, this system can be written as
\[ U_t = U_{xxx} + UU_x \]
\[ V_t = V_{xxx} + \frac{1}{2}(UV)_x \]
where \[U = c_1u + 2c_2v \text{ and } V = c_1u - 2c_2v.\] For a given solution \(U\) of the KdV equations, \(V\) is obtained by solving the linear equation. This result is consistent with that given in Svinolupov (1994).

ACKNOWLEDGMENTS

The author would like to thank Prof. Metin Gürses for stimulating discussions and useful comments. The research reported in this paper was supported in part by the Scientific and Technical Research Council of Turkey.

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