Symmetry, Singularity and Integrability: The final question?

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1 Introduction

Two concepts of integrability of ordinary differential equations.

- An ordinary differential equation is integrable if it has an analytic function as a solution, a criterion advanced by Poincaré.
- Invariance of the equation under infinitesimal transformations generated by differential operators which have become known as symmetries. The possession of a suitable number of these symmetries was sufficient to reduce the solution of the differential equation to a succession of quadratures.
- Painlevé and Lie.

2 Singularity Analysis

Leading order behaviour and next to leading order behaviour.

Painlevé Test: original form

Laurent expansion (RPS)

$$y = \sum_{i=0}^{\infty} a_i \chi^{-p+i}, \tag{2.1}$$

Painlevé Test: LPS

$$y = \sum_{i=0}^{\infty} a_i \chi^{-p-i}, \tag{2.2}$$

Example

$$y'' + 3yy' + y^3 = 0 \tag{2.3}$$

Leading order behaviour

$$y = \alpha\chi^{-p}. \tag{2.4}$$

$$\alpha(-p)(-p-1)\chi^{-p-2} + 3\alpha^2(-p)\chi^{-2p-1} + \alpha^3\chi^{-3p} = 0 \tag{2.5}$$

$p = 1$ and α is given as

$$\alpha^2 - 3\alpha + 2 = 0 \quad \Leftrightarrow \quad \alpha = 1, 2. \tag{2.6}$$

Resonances

$$y = \alpha\chi^{-1} + \mu\chi^{r-1}, \tag{2.7}$$

$$\begin{aligned} & r^2 + 3(\alpha-1)r + 3\alpha^2 - 6\alpha + 2 = 0 \\ \Leftrightarrow \quad & r^2 + 3(\alpha-1)r + \alpha(2\alpha-3) = 0 \end{aligned} \tag{2.8}$$

when (2.6) is taken into account. Hence

$$r = -\alpha, 3 - 2\mu = -1, 1;\ -2, -1 \tag{2.9}$$

Solution

$$y = \frac{2Ax + 2B}{Ax^2 + 2Bx + C}. \tag{2.10}$$

Both Left and Right Painlevé Series are found.

The analysis is representation dependent.

$$Y = \frac{a(x)y + b(x)}{c(x)y + d(x)}, \quad ad - bc \neq 0, \tag{2.11}$$

3 Lie Symmetries

$$\bar{x} = x + \varepsilon\xi \qquad \bar{y} = y + \varepsilon\eta \tag{3.1}$$

$$\Gamma = \xi\partial_x + \eta\partial_y \tag{3.2}$$

$$\bar{x} = (1 + \varepsilon\Gamma)\, x \qquad \bar{y} = (1 + \varepsilon\Gamma)\, y \tag{3.3}$$

$$\bar{f} = (1 + \varepsilon\Gamma)\, f(x, y) = f(x, y) + \varepsilon\left(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}\right). \tag{3.4}$$

Infinitesimal transformation induced in a derivative

$$\begin{aligned}\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} &= \frac{\mathrm{d}(y + \varepsilon\eta)}{\mathrm{d}(x + \varepsilon\xi)} \\ &= \frac{\mathrm{d}y + \varepsilon\mathrm{d}\eta}{\mathrm{d}x + \varepsilon\mathrm{d}\xi} \\ &= \frac{y' + \varepsilon\eta'}{1 + \varepsilon\xi'} \\ &= y' + \varepsilon\left(\eta' - y'\xi'\right)\end{aligned} \tag{3.5}$$

for the transformation induced in the first derivative.

$$\frac{\mathrm{d}^2\bar{y}}{\mathrm{d}\bar{x}^2} = y'' + \varepsilon\left(\eta'' - 2y''\xi' - y'\xi''\right) \tag{3.6}$$

$$\frac{\mathrm{d}^3\bar{y}}{\mathrm{d}\bar{x}^3} = y''' + \varepsilon\left(\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi'''\right) \tag{3.7}$$

etc.

3.1 Functions and equations

$$
\begin{aligned}
\Gamma^{[1]} &= \xi\partial_x + \eta\partial_y + (\eta' - y'\xi')\,\partial_{y'} \\
\Gamma^{[2]} &= \Gamma^{[1]} + (\eta'' - 2y''\xi' - y'\xi'')\,\partial_{y''} \\
\Gamma^{[3]} &= \Gamma^{[2]} + (\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi''')\,\partial_{y'''} \\
&\vdots \\
\Gamma^{[n]} &= \Gamma^{[n-1]} + \left(\eta^{(n)} - \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} y^{(n+1-i)}\xi^{(i)}\right)
\end{aligned}
\tag{3.8}
$$

Function A function, $f\left(x, y, y', \ldots, y^{(n)}\right)$, possesses a symmetry Γ if

$$\Gamma^{[n]} f = 0, \tag{3.9}$$

Equation

$$E\left(x, y, y', \ldots, y^{(n)}\right) = 0, \tag{3.10}$$

possesses a symmetry Γ if

$$\Gamma^{[n]} E_{|_{E=0}} = 0, \tag{3.11}$$

Types of symmetry

4 Selected Examples

4.1 The beloved equation

$$y'' + 3yy' + y^3 = 0,$$

possesses both Left Painlevé Series and Right Painlevé Series.
Riccati transformation

$$y = \alpha\frac{\omega'}{\omega}, \tag{4.1}$$

$$\begin{aligned} \alpha\left(\frac{\omega'''}{\omega} - \frac{3\omega'\omega''}{\omega^2} + \frac{2\omega'^3}{\omega^3}\right) + 3\alpha^2\frac{\omega'}{\omega}\left(\frac{\omega''}{\omega} - \frac{\omega'^2}{\omega^2}\right) + \alpha^3\frac{\omega'^3}{\omega^3} &= 0 \\ \Leftrightarrow \quad \frac{\omega'''}{\omega} + 3(\alpha-1)\frac{\omega'\omega''}{\omega^2} + \left(\alpha^2 - 3\alpha + 2\right)\frac{\omega'^3}{\omega^3} &= 0. \qquad (4.2) \end{aligned}$$

$\alpha = 1$

$$\omega''' = 0 \tag{4.3}$$

The coefficient of ω'^3/ω^3 is the same as for the Painlevé analysis. $\alpha = 1$ leads the RPS and $\alpha = 2$ to the LPS.

$$\omega\omega''' + 3\omega'\omega'' = 0 \tag{4.4}$$

integrating factor ω^2

$$\omega'' = \frac{K}{\omega^3}, \tag{4.5}$$

Ermakov-Pinney equation

The beloved equation possesses eight Lie point symmetries and is linearisable to $Y'' = 0$ by a point transformation.

The beloved equation has it all, the Painlevé Property with both Left Painlevé Series and Right Painlevé Series and with eight Lie point symmetries is linearisable by means of a point transformation.

Can one posit a proposition?

4.2 A theorem breaker

$$y'' = \frac{y'^2}{y} + f(x)yy' + f'(x)y^2 \tag{4.6}$$

is one of the 50 classes of second-order ordinary differential equations of normal form possessing the Painlevé Property.

(4.6) is devoid of Lie point symmetries for a general analytic function $f(x)$ even though it possesses the Painlevé Property, has a trivially obtained invariant and is easily reduced to quadratures.

Generalised Riccati transformation

$$y = -\frac{\omega'}{f(x)\omega} \tag{4.7}$$

$$\omega'\omega''' - \omega''^2 - \omega'^2\left\{\frac{f''}{f} - \frac{f'^2}{f^2}\right\} = 0 \tag{4.8}$$

Also has the obvious symmetry ∂_ω independently of the nature of the function $f(x)$. Suitable of ∂_ω are

$$X = x \quad \text{and} \quad Y = \log\left\{\frac{\omega'}{f}\right\} \tag{4.9}$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = 0 \tag{4.10}$$

4.3 The generalised Kummer-Schwarz equation

Is there a connection between the two properties of a solution in terms of an analytic function and a connection to a linear system?

$$y'y''' - ny''^2 = 0 \tag{4.11}$$

General n
∂_x, ∂_y, $x\partial_x$ and $y\partial_y$ with $A_2 \oplus A_2$.
For $n = 3/2$
additionally $x^2\partial_x$ and $y^2\partial_y$ with $sl(2,R) \oplus sl(2,R)$, but ten contact symmetries with $sp(5)$.

The Kummer-Schwarz equation possesses the Painlevé Property with resonances at $r = -1, 0, 1$, *ie* there is a Right Painlevé Series but no Left Painlevé Series.

For the Kummer-Schwarz equation the leading exponent is $(n-2)/(n-1)$, $n \neq (m-2)/(m-1)$, where m is a nonnegative integer, and the Kowalevski exponents are $r = 0, -1$ and $-(n-2)/(n-1)$. For rational n in $(1, 2)$ (4.11) possesses a Right Painlevé Series for what is generically the weak Painlevé Property. For rational n outside this interval there is a Left Painlevé Series.

The generalised Kummer-Schwarz equation may be linearised to

$$Y''' = 0 \tag{4.12}$$

for all $n \neq 1$ by means of the nonlocal transformation

$$Y = \int y'^{-n+1} \mathrm{d}x \quad X = x. \tag{4.13}$$

(In the case $n = 1$ the integrand is $\log y'$.)

In the case of the generalised Kummer-Schwarz equation the nature of the transformation intrudes itself. The solution of (4.11) possesses the Painlevé Property, respectively the weak Painlevé Property, if the transformation possesses at most a polelike or branch point singularity.

5 A Conjecture