Orbits and Nonlocal Symmetries

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1 The Kepler Problem

Aristarchos of Samos
Aristotle
Nicholas Copernicus
Tycho Brahe
Johannes Kepler
Issac Newton
1.1 Conserved Vectors and Orbit

\[ \ddot{r} + \frac{\mu}{r^3} r = 0, \]  
(1.1)

Angular momentum

\[ L := r \times \dot{r} \]  
(1.2)

Laplace-Runge-Lenz (Ermanno-Bernoulli)

\[ J := \dot{r} \times L - \mu \hat{r}, \]  
(1.3)

Hamilton

\[ K := \dot{r} - \frac{\mu}{L} \dot{\theta} \]

The Triad \( J = K \times L \).

A scalar

\[ E = \frac{1}{2} \dot{r} \cdot \dot{r} - \frac{\mu}{r} \]

A relationship

\[ J^2 = 2L^2E + \mu^2 \]

The orbit

\[ rJ \cos \theta = L^2 - \mu r \]

\[ \Leftrightarrow \quad r = \frac{L^2}{\mu + J \cos \theta}. \]  
(1.4)
2 Lie symmetries

1994 and 1995 Krause: The complete symmetry group of a differential equation is the group associated with the set of symmetries, be they point, contact, generalised or nonlocal, required to specify the equation or system completely.

Specifically he required that the elements of the group have the two properties that

the manifold of solutions be an homogeneous space of the group and

the group be specific to the system, *ie* no other system admits it.

Subsequently a requirement of minimality was added after Andriopoulos *et al* showed that not only was the group not unique but also the dimensionality could vary.
The elements of the five-dimensional algebra are

\[
\begin{align*}
X_1 &= \partial_t \\
X_2 &= t\partial_t + \frac{2}{3}r\partial_r \\
X_3 &= x_2\partial_{x_3} - x_3\partial_{x_2} \\
X_4 &= x_3\partial_{x_1} - x_1\partial_{x_3} \\
X_5 &= x_1\partial_{x_2} - x_2\partial_{x_1}
\end{align*}
\]

in which \(x_1, x_2\) and \(x_3\) are the usual cartesian components of the position vector \(r\) of magnitude \(r\). These five point symmetries of the equation of motion are insufficient to specify the equation completely. To overcome the deficiency in the number of symmetries Krause introduced a nonlocal symmetry of specific structure defined by

\[
Y = \left[\int \xi(t, x_1, \ldots, x_N)dt\right] \partial_t + \sum_{i=1}^{N} \eta_i(t, x_1, \ldots, x_N)\partial_{x_i}
\]

and obtained three symmetries of this form, \textit{videlicet}

\[
\begin{align*}
Y_1 &= 2 \left(\int x_1 dt\right) \partial_t + x_1 r\partial_r \\
Y_2 &= 2 \left(\int x_2 dt\right) \partial_t + x_2 r\partial_r \\
Y_1 &= 2 \left(\int x_3 dt\right) \partial_t + x_3 r\partial_r,
\end{align*}
\]

which has the compact form

\[
Y = 2 \left(\int r dt\right) \partial_t + rr\partial_r,
\]

in which \(r^2 = x_1^2 + x_2^2 + x_3^2\), for the Kepler Problem.
Nucci and Leach: more nonlocal symmetries of the structure adopted by Krause obtainable as point symmetries by the reduction method of Nucci.

These additional nonlocal symmetries have a place in the discussion of the complete symmetry group.

There are three equivalent representations of the complete symmetry group.

All symmetries are found by point methods.

There is no need to use the method of reduction of order.

Mahomed and Leach: first integrals of the simple harmonic oscillator, $\ddot{x} + x = 0$,

\begin{align*}
I_1 &= x \cos t - \dot{x} \sin t \\
I_2 &= x \sin t + \dot{x} \cos t \\
I_3 &= \frac{x \cos t - \dot{x} \sin t}{x \sin t + \dot{x} \cos t}
\end{align*}

(2.5)

each possessed three Lie point symmetries with the same algebra.

Andriopoulos et al: this algebra provided the elements of the complete symmetry group of the equation of motion for the simple harmonic oscillator.
The three equivalent representations lead to three equivalent representations of the complete symmetry group of the Kepler Problem.

The explicit cartesian components of $\mathbf{J}$ are

$$J_x = (r^3 \dot{\theta}^2 - \mu) \cos \theta + r^2 \dot{r} \dot{\theta} \sin \theta \quad (2.6)$$

$$J_y = (r^3 \dot{\theta}^2 - \mu) \sin \theta - r^2 \dot{r} \dot{\theta} \cos \theta. \quad (2.7)$$

From the combination

$$J_x \pm iJ_y = (r^3 \dot{\theta}^2 - \mu) e^{\pm i\theta} \pm ir^2 \dot{r} \dot{\theta} e^{\pm i\theta}$$

$$= L^2 \left( \frac{1}{r} - \frac{\mu}{L^2} \pm i \frac{\dot{r}}{r^2 \dot{\theta}} \right)$$

$$= \mp iL^2 \left( \left( \frac{1}{r} \right)' \pm i \left( \frac{1}{r} - \frac{\mu}{L^2} \right) \right) e^{\pm i\theta}, \quad (2.8)$$

where $L := r^2 \dot{\theta}$ and the prime denotes differentiation with respect to $\theta$, we define the Ermanno-Bernoulli constants as

$$J_\pm = (v_1' \pm iv_1) e^{\pm i\theta}, \quad (2.9)$$

where $v_1 = 1/r - \mu/L^2$. The radial equation of motion is

$$v_1'' + v_1 = 0. \quad (2.10)$$

$$J = L^2 (v_1 \hat{r} + v_1' \hat{\theta}). \quad (2.11)$$
3 The three representations of the complete symmetry group of the Kepler Problem

In terms of \( v_1, v_2 \) and \( \theta \) the two-dimensional Kepler Problem is

\[
\begin{align*}
    v_1'' + v_1 &= 0 \quad \text{(3.1)} \\
    v_2' &= 0. \quad \text{(3.2)}
\end{align*}
\]

The Lie point symmetries are

\[
\begin{align*}
    \Gamma_1 &= \partial_{v_2} \quad \Gamma_4 \pm = e^{\pm i\theta} \partial_{v_1} \\
    \Gamma_2 &= \partial_{\theta} \quad \Gamma_5 \pm = e^{\pm 2i\theta} (\partial_{\theta} \pm iv_1 \partial_{v_1}) \quad \text{(3.3)} \\
    \Gamma_3 &= v_1 \partial_{v_1} \quad \Gamma_6 \pm = e^{\pm i\theta} \left( v_1 \partial_{\theta} \pm iv_1^2 \partial_{v_1} \right).
\end{align*}
\]

These plus \( \partial_t \) are the symmetries for finding the complete symmetry group of the Kepler Problem.

The invariants and symmetries are

\[
\begin{align*}
    I_A &= (v_1 + iv_1') e^{i\theta} \quad \left\{ \begin{array}{l}
    A_1 = e^{i\theta} \partial_{v_1} \\
    A_2 = \partial_{\theta} - iv_1 \partial_{v_1} \\
    A_3 = e^{2i\theta} (\partial_{\theta} + iv \partial_{v_1})
\end{array} \right. \quad \text{(3.4)} \\
    I_B &= (v_1 - iv_1') e^{-i\theta} \quad \left\{ \begin{array}{l}
    B_1 = e^{-i\theta} \partial_{v_1} \\
    B_2 = \partial_{\theta} + iv_1 \partial_{v_1} \\
    B_3 = e^{-2i\theta} (\partial_{\theta} - iv \partial_{v_1})
\end{array} \right. \quad \text{(3.5)} \\
    I_C &= \frac{v_1 + iv_1'}{v_1 - iv_1'} e^{2i\theta} \quad \left\{ \begin{array}{l}
    C_1 = v_1 \partial_{v_1} \\
    C_2 \pm = e^{\pm i\theta} (v_1 \partial_{\theta} \pm iv_1^2 \partial_{v_1}) \end{array} \right. \quad \text{(3.6)}
\end{align*}
\]
Proposition: The complete symmetry group of the two-dimensional Kepler Problem has three (equivalent) representations given by the sets $A$, $B$ or $C$ plus $\partial_t$.

A $(t, r, \theta)$ symmetry becomes a $(\theta, v_1, v_2)$ symmetry according to

$$\tau \partial_t + \eta \partial_r + \zeta \partial_\theta \quad \longrightarrow \quad \zeta \partial_\theta + \Omega \partial_{v_1} + \Sigma \partial_{v_2},$$

(3.7)

where

$$v_2 = r^2 \dot{\theta}, \quad v_1 = \frac{1}{r} - \frac{\mu}{v_2^2}$$

$$\Sigma = 2 \eta r \dot{\theta} + r^2 \left( \zeta - \dot{\theta} \tau \right)$$

$$\Omega = -\frac{\eta}{r^2} + \frac{2 \mu}{v_2^3} \Sigma.$$

In $(t, r, \theta)$ coordinates we obtain

$$A_1 = \left( 2 \int r e^{i\theta} \frac{dt}{dt} \right) \partial_t + r^2 e^{i\theta} \partial_r$$

$$A_2 = 2 \left( t - \frac{\mu}{L^2} \int r \frac{dt}{dt} \right) + r \left( 1 - \frac{\mu r}{L^2} \right) \partial_r - i \partial_\theta$$

(3.9)

$$A_3 = \left( \frac{2i \mu}{L^2} \int r e^{2i\theta} \frac{dt}{dt} \right) \partial_t - i r \left( 1 - \frac{\mu r}{L^2} \right) e^{2i\theta} \partial_r + e^{2i\theta} \partial_\theta.$$
In the case of negative energy as generalised symmetries we have

\begin{align}
A_1 &= \left[ \frac{ir^2 L}{J} + \frac{\mu r^2 \sin \theta}{2(-2E)L} + \frac{r(\mu^2 + 2J^2) \sin \theta}{2(-2E)^2 L} \right. \\
&\quad - \frac{6\mu J}{(-2E)^{5/2}} \arctan \left( \frac{\mu - J}{\mu + J} \right)^{1/2} \tan \frac{\theta}{2} \left. \right] \partial_t + r^2 e^{i\theta} \partial_r \\
A_2 &= 2 \left[ t - \frac{\mu}{L^2} \left( \frac{r^2 J \sin \theta}{4EL} - \frac{3\mu r J \sin \theta}{8E^2 L} \right) \right. \\
&\quad + \frac{2\mu^2 + J^2}{(-2E)^{5/2}} \arctan \left( \frac{\mu - J}{\mu + J} \right)^{1/2} \tan \frac{\theta}{2} \left. \right] \partial_t \\
&\quad + r \left( 1 - \frac{\mu r}{L^2} \right) \partial_r + \partial_{\theta} \\
A_3 &= \frac{2i\mu}{L^2} \left[ \frac{(2\mu^2 - J^2) \mu r^2 \sin \theta}{2J^2(-2E)L} \right. \\
&\quad + \frac{2\mu^4 + 3\mu^2 J^2 - 2J^4 - 8\mu J^3 + 8\mu^3 J}{2J^2 L(-2E)^2} r \sin \theta \\
&\quad - \frac{4\mu^4 + 12\mu^3 J - 6\mu J^3 - 4J^4}{J^2(-2E)^{5/2}} \arctan \sqrt{\frac{\mu - J}{\mu + J} \tan \frac{1}{2} \theta} \left. \right] \partial_t \\
&\quad - ir \left( 1 - \frac{\mu r}{L^2} \right) e^{2i\theta} \partial_r + e^{2i\theta} \partial_{\theta}.
\end{align}
The Kepler Problem in three dimensions can be treated similarly, but at greater effort.

The three-dimensional Kepler Problem is completely specified by the six-dimensional algebra \( A_1 \oplus \{ A_1 \oplus_s \{ 2A_1 \oplus 2A_1 \} \} \).

Krause reported that eight symmetries were needed to specify the three-dimensional Kepler Problem. Evidently he did not make the correct choice of symmetries.
4 Other systems with a conserved vector of Laplace-Runge-Lenz or Hamilton type

4.1 The model equation $\ddot{r} + f r = 0$

The central force equation

$$\ddot{r} + v(\theta)\dot{\theta} \frac{\dot{r}}{r} = 0,$$  \hspace{1cm} (4.1)

Hamilton and Laplace-Runge-Lenz vectors

$$K = \dot{r} + z'(\theta)\dot{r} - z(\theta)\dot{\theta}$$  \hspace{1cm} (4.2)

$$J = \dot{r} \times \dot{L} - z(\theta)\dot{r} - z'(\theta)\dot{\theta},$$  \hspace{1cm} (4.3)

where

$$z''(\theta) + z(\theta) = v(\theta)$$  \hspace{1cm} (4.4)

subject to the initial conditions $z(\theta_0) = 0$ and $z'(\theta_0) = 0$.

4.2 Extension to nonautonomous systems

$$\ddot{r} + \frac{Lv(\theta)}{g(t)r^3} \dot{r} - \frac{\ddot{g}(t)}{g(t)} r = 0,$$  \hspace{1cm} (4.5)

where $L$ is treated as a constant in the equation of motion, although it is expressed as $r^2\dot{\theta}$ during calculations.

Hamilton-like vector

$$K = g\dot{r} - \dot{g}r + z'(\theta)\dot{r} - z(\theta)\dot{\theta},$$  \hspace{1cm} (4.6)
Laplace-Runge-Lenz vector

\[ J = (g \dot{r} - \dot{g}r) \times \hat{L} - z(\theta) \hat{r} - z'(\theta) \hat{\theta}. \] (4.7)

4.3 Vector conservation laws for the equation of motion \( \ddot{r} + g \dot{r} + h \dot{\theta} = 0 \)

\[ \ddot{r} + \left[ \frac{U''(\theta) + U(\theta)}{r^2} + \frac{2V'(\theta)}{r^2} \right] \hat{r} + \frac{V(\theta)}{r^2} \hat{\theta} = 0 \] (4.8)

has the Laplace-Runge-Lenz vector

\[ J = \dot{r} \times L - U(\theta) \hat{r} - \left[ U'(\theta) + 2r^2 V(\theta) \right] \hat{\theta}. \] (4.9)

and the Hamilton’s vector

\[ K = \hat{L} \times J = L \dot{r} - U(\theta) \hat{\theta} + \left[ U'(\theta) + 2r^2 V(\theta) \right] \hat{r}. \] (4.10)

4.4 The Kepler problem with ‘drag’

The Danby problem is

\[ \ddot{r} + \frac{\alpha \dot{r}}{r^2} + \frac{\mu r}{r^3} = 0. \] (4.11)

The conserved vectors are

\[ K = \frac{\dot{r}}{L} + \frac{\mu}{\alpha^2} \left( \frac{1}{u} - \sin u \text{Ci}(u) + \cos u \text{Si}(u) \right) \hat{r} \]
\[ J = \frac{\dot{\mathbf{r}} \times \dot{\mathbf{L}}}{L} + \frac{\mu}{\alpha^2} (\sin \mu \text{si}(\mu) + \cos \mu \text{Ci}(\mu)) \dot{\mathbf{r}} \]

\[ -\frac{\mu}{\alpha^2} \left( \frac{\dot{\mathbf{r}}_{\theta_0}}{u_0} + \text{si}(u_0)\dot{r}_{k/\alpha} + \text{Ci}(u_0)\dot{\theta}_{k/\alpha} \right) \]

\[ (4.12) \]

where the sine and cosine integrals are given by

\[ \text{si}(x) = -\frac{1}{2} \pi + \int_0^x \frac{\sin t}{t} \, dt \]

\[ (4.14) \]

\[ \text{Ci}(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t} \, dt \]

\[ (4.15) \]

and \( \gamma = 0.57721566490 \ldots \) is Euler’s constant.

5 \hspace{1em} \textbf{Force laws admitting Keplerian orbits}

\[ \ddot{\mathbf{r}} - \frac{1}{2} \left( \frac{g}{g} + 3 \frac{\dot{r}}{r} \right) \dot{\mathbf{r}} + g \mathbf{r} = 0. \]

\[ (5.1) \]

\[ \mathbf{K} = \frac{\mathbf{r}'}{L_t} - A (\dot{\theta} - \dot{\theta}_{\theta_0}) \]

\[ \mathbf{J} = \frac{\mathbf{r}' \times L_t}{L_t} - A (\mathbf{r} - \dot{\mathbf{r}}_{\theta_0}), \]

\[ (5.2) \]
where \( \hat{r}_{\theta_0} = \hat{i} \cos \theta_0 + \hat{j} \sin \theta_0 \) and \( \hat{\theta}_{\theta_0} = -\hat{i} \sin \theta_0 + \hat{j} \cos \theta_0 \).

5.1 A conserved Laplace-Runge-Lenz vector for

\[
\ddot{r} + f \dot{L} + g \dot{r} = 0
\]

possesses the conserved Laplace-Runge-Lenz vector

\[
J = \dot{r} \times L - h(r) L - k \dot{r}.
\]

5.2 The classical MICZ problem

\[
\ddot{r} + \frac{\lambda}{r^3} \dot{L} + \left( \frac{\mu}{r^2} - \frac{\lambda^2}{r^3} \right) \dot{r} = 0,
\]

Poincare’s vector

\[
P = L - \lambda \dot{r},
\]

Laplace-Runge-Lenz vector

\[
J = \dot{r} \times L + \frac{\lambda}{r} L - \mu \dot{r}.
\]

Hamilton’s vector

\[
K = J \times P = (\dot{r} \times P) \times P - \frac{\mu}{r} r \times P.
\]
5.3 A truly three-dimensional motion

\[ \ddot{\mathbf{r}} + \frac{h'}{r} \mathbf{L} + \left( hh' + \frac{k}{r^2} \right) \hat{\mathbf{r}} = 0 \]  \hspace{1cm} (5.9)

has the Laplace-Runge-Lenz vector

\[ \mathbf{J} = \dot{\mathbf{r}} \times \mathbf{L} - h(r) \mathbf{L} - k \mathbf{r}. \]  \hspace{1cm} (5.10)
6 Conclusion

There are three representations of the complete symmetry group for the Kepler Problem.

These three representations follow directly from the three representations of the complete symmetry group for the simple harmonic oscillator.

Nucci and Leach showed that all the related problems possessing a conserved vector of Laplace-Runge-Lenz type may be reduced to the simple harmonic oscillator and we may infer that these systems also have three representations for their complete symmetry groups.

One expects that the algebra be the same.

Given that the algebra is the same, there should exist a transformation of coordinates between any two of these systems.