Money, Medicine and Motion
a symmetric approach

PGL Leach

School of Mathematical and Statistical Sciences
Howard College, University of KwaZulu-Natal
Durban 4041, Republic of South Africa

in conjunction with

S Moyo, V Naicker, K Andriopoulos and AN Yannacopoulos

Department of Physics

METU, Ankara
18:11:2004

Supported by TUBITAK through a NATO-D grant

1 Motion

Noether symmetries for a classical Lagrangian

$$L = \tfrac{1}{2}\dot{x} - V(t,x). \tag{1.1}$$

V	*no sym*	*algebra*
$V(t,x)$	0	$--$
$V(x)$	1	A_1
$\omega^2 x^2 + \dfrac{h^2}{x^2}$	3	$sl(2,R)$
$\omega^2 x^2$	5	$sl(2,R) \oplus_s 2A_1$
$\mu^2 x$	5	$sl(2,R) \oplus_s 2A_1$
μ^2	5	$sl(2,R) \oplus_s W.$

Lie point symmetries of the corresponding evolution equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + V(t,x)u = 0, \tag{1.2}$$

V	no sym	algebra
$V(t,x)$	$0+1+\infty$	$A_1 \oplus_s \infty A_1$
$V(x)$	$1+1+\infty$	$2A_1 \oplus_s \infty A_1$
$\omega^2 x^2 + \dfrac{h^2}{x^2}$	$3+1+\infty$	$\{A_1 \oplus sl(2,R)\} \oplus_s \infty A_1$
$\mu^2 + \dfrac{h^2}{x^2}$	$3+1+\infty$	$\{A_1 \oplus sl(2,R)\} \oplus_s \infty A_1$
$\omega^2 x^2$	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$
$\mu^2 x$	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$
μ^2	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1,$

(1.3)

where W is the Heisenberg-Weyl algebra with

$$[\Sigma_1, \Sigma_2]_{LB} = 0,\ [\Sigma_1, \Sigma_3]_{LB} = 0,\ [\Sigma_2, \Sigma_3]_{LB} = \Sigma_1$$

.

2 Money

2.1 Terminal condition $u(T,x) = U$

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \tfrac{1}{2}\omega^2 x^2 u &= 0 \\ u(T,x) &= U. \end{aligned}$$

$$\begin{aligned} \Gamma_1 &= f(t,x)\partial_u \\ \Gamma_2 &= u\partial_u \\ \Gamma_3 &= \partial_t \\ \Gamma_4 &= \mathrm{e}^{2\omega t}\left(\partial_t + \omega x\partial_x + (\omega^2 x^2 - \tfrac{1}{2}\omega)u\partial_u\right) \\ \Gamma_5 &= \mathrm{e}^{-2\omega t}\left(\partial_t - \omega x\partial_x + (\omega^2 x^2 + \tfrac{1}{2}\omega)u\partial_u\right) \\ \Gamma_6 &= \mathrm{e}^{\omega t}\left(\partial_x + \omega x u\partial_u\right) \\ \Gamma_7 &= \mathrm{e}^{-\omega t}\left(\partial_x - \omega x u\partial_u\right) \end{aligned}$$

$$\begin{aligned} a_3 + a_4\mathrm{e}^{2\omega T} + a_5\mathrm{e}^{-2\omega T} &= 0 \\ a_2 + a_4\mathrm{e}^{2\omega T}\left(\omega^2 x^2 - \tfrac{1}{2}\omega\right) + a_5\mathrm{e}^{-2\omega T}\left(\omega^2 x^2 + \tfrac{1}{2}\omega\right) &= 0, \end{aligned}$$

$$\begin{aligned} a_3 + a_4\mathrm{e}^{2\omega T} + a_5\mathrm{e}^{-2\omega T} &= 0 \\ a_2 + a_4\mathrm{e}^{2\omega T}\left(\omega^2 x^2 - \tfrac{1}{2}\omega\right) + a_5\mathrm{e}^{-2\omega T}\left(\omega^2 x^2 + \tfrac{1}{2}\omega\right) + a_6\mathrm{e}^{\omega T}(\omega x) + a_7\mathrm{e}^{-\omega T}(-\omega x) &= 0 \end{aligned}$$

$$\begin{aligned} \Sigma_1 &= \cosh\omega(t-T)\partial_x + \omega x u\sinh\omega(t-T) \\ \Sigma_2 &= \sinh 2\omega(t-T)\partial_t + \omega x\cosh 2\omega(t-T)\partial_x + \left(\tfrac{1}{2}\omega(1-\cosh 2\omega(t-T)) + \omega^2 x^2 \sinh 2\omega(t-T \right. \end{aligned}$$

Lie Bracket $[\Sigma_1, \Sigma_2]_{LB} = \omega\Sigma_1$.

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\cosh\omega(t-T)} = \frac{\mathrm{d}u}{\omega x u\sinh\omega(t-T)}$$

$$t \text{ and } u\left[-\tfrac{1}{2}\omega x^2\tanh\omega(t-T)\right] \qquad u = f(t)\left[\tfrac{1}{2}\omega x^2\tanh\omega(t-T)\right]$$

$$u(t,x) = U\cosh^{-1/2}\omega(t-T)\left[\tfrac{1}{2}\omega x^2\tanh\omega(t-T)\right]$$

2.2 Application to a mean variance hedging equation

$$\frac{\partial J}{\partial t} + a\frac{\partial J}{\partial y} + \tfrac{1}{2}b^2\frac{\partial^2 J}{\partial y^2} - \tfrac{1}{2}\left(\frac{\partial J}{\partial y}\right)^2 + \nu(y) = 0 \tag{2.1}$$

with the terminal condition $J(T, y) = 0$
$\nu(x) = \mu^2$ for which the symmetries are

$$\begin{aligned}
\Gamma_1 &= f(t,x)\exp[J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= t\partial_t + \tfrac{1}{2}(x+at)\partial_x + \mu^2 t\partial_J \\
\Gamma_5 &= t^2\partial_t + tx\partial_x + \tfrac{1}{2}\left(b^2 t - 2\mu^2 tx - (x-at)^2\right)\partial_J \\
\Gamma_6 &= \partial_x \\
\Gamma_7 &= t\partial_x - (x-at)\partial_J,
\end{aligned} \tag{2.2}$$

where $f(t, x)$ is a solution of

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \frac{\mu^2}{b^2}f = 0, \tag{2.3}$$

Also $\nu(x) = \mu^2 x$ **and** $\nu(x) = \frac{1}{2}\omega^2 x^2$

$$J = \mu^2(T-t), \tag{2.4}$$

$$J(t,x) = \tfrac{1}{2}a\mu^2(t-T)^2 + \tfrac{1}{6}\mu^4(t-T)^3 - \mu^2(t-T)x \tag{2.5}$$

and

$$J(t,x) = a\left[1 - \text{sech}\,\omega(t-T)\right] + \tfrac{1}{2\omega}\left(a^2 - \omega^2 x^2\right)\tanh\omega(t-T) + \tfrac{1}{2}b^2\log\cosh\omega(t-T) - \tfrac{1}{2}a^2(t-T) \tag{2.6}$$

2.3 Extension to a time-dependent $\nu(t, x)$

$\nu(t, x) = \mu^2(t)$

$$
\begin{aligned}
\Gamma_1 &= f(t,x)\exp[J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_x \\
\Gamma_4 &= \partial_t - \mu^2(t)\partial_J \\
\Gamma_5 &= t\partial_x + (at - x)\partial_J \\
\Gamma_6 &= 2t\partial_t + (at + x)\partial_x - 2t\mu^2(t)\partial_J \\
\Gamma_7 &= 2t^2\partial_t + 2tx\partial_x + \left(b^2 t - 2t^2\mu^2(t) - (x - at)^2\right)\partial_J,
\end{aligned}
$$

where $f(t, x)$ is a solution of

$$
\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \frac{\mu^2(t)}{b^2}f = 0. \tag{2.7}
$$

$$
\begin{aligned}
a_4 + 2Ta_6 + 2Ta_7 &= 0 \\
a_2 - \mu^2(T)a_4 + (aT - x)a_5 - 2T\mu^2(T)a_6 & \\
+(2aTx - a^2T^2 + b^2T - x^2 - 2T^2\mu^2(T))a_7 &= 0.
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_1 &= \partial_x \\
\Sigma_2 &= 2(t - T)\left(\partial_t - \mu^2(t)_J\right).
\end{aligned}
$$

$$
J = \int_{t_0}^{T}\mu(s)^2\mathrm{d}s - \int_{t_0}^{t}\mu(s)^2\mathrm{d}s \tag{2.8}
$$

3 Medicine: The Burgess equation

A model of the spread of aggressive brain cancers such as glioblastoma multiforme:

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + pn(r,t) - kn(r,t) \tag{3.1}$$

in which $n(r,t)$ is the concentration of tumour cells at location r at time t, D is the diffusion coefficient, *ie* a measure of the areal speed of the invading glioblastoma cells, p is the assumed constant rate of reproduction of the glioblastoma cells and k the killing rate of the same cells.

$$\tau = 2Dt, \quad \phi(r,\tau) = rn(r,t), \quad w = \frac{k-p}{2D} \tag{3.2}$$

(3.1) becomes

$$-\frac{\partial \phi}{\partial \tau} = -\tfrac{1}{2}\frac{\partial^2 \phi}{\partial r^2} + w\phi. \tag{3.3}$$

w can be $w(r,t)$ without affecting (3.3).

Other aspects of the model.

4 Symmetry-based solutions for the Burgess equation

The nontrivial Lie point symmetries of the Burgess equation

$$2\frac{\partial \phi}{\partial \tau} - \frac{\partial^2 \phi}{\partial \rho^2} + 2w\phi = 0$$

are

$$\begin{aligned}
\Gamma_1 &= \partial_\tau \\
\Gamma_2 &= 2\tau\partial_\tau + \rho\partial_\rho - 2w\rho\phi\partial_\phi \\
\Gamma_3 &= 2\tau^2\partial_\tau + 2\tau\rho\partial_\rho - \left[2w\tau^2 + \tau + \rho^2\right]\phi\partial_\phi \\
\Gamma_4 &= \partial_\rho \\
\Gamma_5 &= \tau\partial_\rho - \rho\phi\partial_\phi.
\end{aligned}$$

We generate solutions as for the time-dependent Schrödinger equation.

The associated Lagrange's system for Γ_4 is

$$\frac{\mathrm{d}\tau}{0} = \frac{\mathrm{d}\rho}{1} = \frac{\mathrm{d}\phi}{0}$$

with the corresponding characteristics $u = \tau$ and $v = \phi$, thus $\phi = f(\tau)$. Equation (3.3) now gives

$$-\dot{f} = wf \Leftrightarrow f(\tau) = c \exp[-w\tau],$$

that is

$$\phi_0 = \exp[-w\tau] \quad \text{and} \quad \Sigma_0 = \phi^{-1} \exp[-w\tau],$$

where Σ_0 is the surface in (τ, ρ, ϕ) space of the basis solution. Obviously ϕ_0 is a particular solution of (3.3). The vehicle for the production of further solutions is Σ_0 and the way is to operate the other solution symmetry, Γ_5, on Σ_0, ie

$$\Gamma_5\Sigma_0 = -\rho\phi(-\phi^{-2} \exp[-w\tau]) = \rho\phi^{-1} \exp[-w\tau].$$

Thus

$$\phi_1 = \rho \exp[-w\tau] \quad \text{and} \quad \Sigma_1 = \rho\phi^{-1} \exp[-w\tau].$$

This procedure results in the construction of an infinite set of particular solutions. We report a few more in order to give a clue for the derivation of the general formula of these symmetries.

$$\begin{aligned}
\phi_2 &= (\tau + \rho^2)\exp[-w\tau] \\
\phi_3 &= \rho(3\tau + \rho^2)\exp[-w\tau] \\
\phi_4 &= (3\tau^2 + 6\tau\rho^2 + \rho^4)\exp[-w\tau] \\
\phi_5 &= \rho(15\tau^2 + 10\tau\rho^2 + \rho^4)\exp[-w\tau] \\
\phi_6 &= (15\tau^3 + 45\tau^2\rho^2 + 15\tau\rho^4 + \rho^6)\exp[-w\tau].
\end{aligned}$$

In general we have

$$\phi_{2n} = \left(\sum_{j=0}^{n} \frac{2^j n!}{(2j)!(n-j)!}\rho^{2j}\tau^{n-j}\right)\exp[-w\tau] \tag{4.1}$$

for the even solutions and

$$\phi_{2n+1} = \left(\rho\sum_{j=0}^{n} \frac{2^j n!}{(n-j)!(2j+1)!}\rho^{2j}\tau^{n-j}\right)\exp[-w\tau] \tag{4.2}$$

for the odd solutions.

What about the reverse procedure?

The associated Lagrange's system for Γ_5 is

$$\frac{\mathrm{d}\tau}{0} = \frac{\mathrm{d}\rho}{\tau} = \frac{\mathrm{d}\phi}{-\rho\phi}$$

so that the two invariants are

$$v = \tau \quad \text{and} \quad w = \phi \exp\left[\tfrac{1}{2}\rho^2/\tau\right].$$

We write

$$\begin{aligned}
\phi &= f(\tau)\exp\left[-\tfrac{1}{2}\rho^2/\tau\right] \\
\frac{\partial\phi}{\partial\tau} &= \left(\dot{f} + \tfrac{1}{2}\frac{\rho^2}{\tau^2}f\right)\exp\left[-\tfrac{1}{2}\rho^2/\tau\right] \\
\frac{\partial^2\phi}{\partial\rho^2} &= f\left(-\frac{1}{\tau} + \frac{\rho^2}{\tau^2}\right)\exp\left[-\tfrac{1}{2}\rho^2/\tau\right].
\end{aligned}$$

Then

$$2\frac{\partial\phi}{\partial\tau} - \frac{\partial^2\phi}{\partial\rho^2} + 2w\phi = 0$$

becomes

$$\begin{aligned}
2\dot{f} + \frac{\rho^2}{\tau^2}f - \left(-\frac{1}{\tau} + \frac{\rho^2}{\tau^2}\right)f + 2wf &= 0 \\
\frac{\dot{f}}{f} = -\tfrac{1}{2}\left(\frac{1}{\tau} + 2w\right) \quad &\Longrightarrow \quad f = \tau^{-1/2}\mathrm{e}^{-w\tau}
\end{aligned}$$

and so we find the 'ground-state' solution

$$\phi_0 = \tau^{-1/2}\exp\left[-w\tau - \tfrac{1}{2}\rho^2/\tau\right].$$

We use $\Gamma_4 = \partial_\rho$ to generate other solutions as before. Some are

$$\begin{aligned}
\phi_1 &= -\rho\tau^{-3/2}\exp\left[-w\tau - \tfrac{1}{2}\rho^2/\tau\right] \\
\phi_2 &= \left(\rho^2 - \tau\right)\tau^{-5/2}\exp\left[-w\tau - \tfrac{1}{2}\rho^2/\tau\right] \\
\phi_3 &= -\left(\rho^3 - 3\rho\tau\right)\tau^{-7/2}\exp\left[-w\tau - \tfrac{1}{2}\rho^2/\tau\right] \\
\phi_4 &= \left(\rho^4 - 6\rho^2\tau + 3\tau^2\right)\tau^{-9/2}\exp\left[-w\tau - \tfrac{1}{2}\rho^2/\tau\right].
\end{aligned}$$

These are not solutions to the Burgess equation!
To obtain them one introduces the transformation

$$\tau = 2Dt, \quad \phi(r,\tau) = rn(r,t), \quad w = \frac{k-p}{2D}.$$

Some solutions of the Burgess equation

Some of the solutions of the Burgess equation obtained by the transformations of solutions generated from the basis solution of Γ_4 are

$$\begin{aligned}
n_0 &= \frac{1}{r}\exp\left[(p-k)t\right] \\
n_1 &= \frac{1}{r^2}\exp\left[(p-k)t\right] \\
n_2 &= \left(r+\frac{2Dt}{r}\right)\exp\left[(p-k)t\right] \\
n_3 &= (6Dt+r^2)\exp\left[(p-k)t\right] \\
&\vdots \\
n_{2n} &= \frac{1}{r}\left(\sum_{j=0}^{n}\frac{2^j n!}{(2j)!(n-j)!}r^{2j}(2Dt)^{n-j}\right)\exp\left[(p-k)t\right] \\
n_{2n+1} &= \left(\sum_{j=0}^{n}\frac{2^j n!}{(2j+1)!(n-j)!}r^{2j}(2Dt)^{n-j}\right)\exp\left[(p-k)t\right].
\end{aligned}$$

5 Other models for the Burgess equation

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + pn(r,t) - kn(r,t) \tag{5.1}$$

which contains a Malthusian term consisting of two parts, both of which were constant. The first term contains the rate of replication of the glioblastoma cells and the second term contains a killing term which could be due to intervention such as chemotherapy.

We could examine the effect of a proliferation rate, p, which is not a constant but which depends upon the radial distance r. We could also examine the effect of a killing term, k, which is a function of time.

The assumption of a constant proliferation rate does not take into account the effect of crowding and is part of the assumption of growth in a uniform and isotropic medium. We can assume that the proliferation rate is proportional to the radial distance, that is

$$p = cr^{\alpha}, \quad \alpha > 0. \tag{5.2}$$

Possible values for α which are mathematically amenable are $\alpha = 1, 2, -2$.

In the case of $\alpha = 1$ the Burgess equation is

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + crn(r,t) - kn(r,t).$$

When $\alpha = 2$, the Burgess equation has the form

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + cr^2n(r,t) - kn(r,t).$$

A third possibility for a nontrivial algebra of Lie point symmetries occurs when $\alpha = -2$, *ie*

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + cr^{-2}n(r,t) - kn(r,t).$$

6 Can there be Chaos?