Commonalities of the Evolution Equations in Chemistry Physics, Medicine and Economics

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1 Mathematical Modelling and Science

Mathematical Modelling is now fashionable and courses – at the postgraduate level – abound.

For some strange reason Mathematical Modelling is perceived to be concerned with the 'new' subjects:
Economics, Medicine, Biology, Epidemiology, Ecology . . . .

Yet Mathematical Modelling started with Physics, in particular Mechanics. One can go back to the times of Plato and Aristotle! Maybe they were not so good at it, but that is another story.

Francis Bacon advocated experiment to test ideas.
Galileo Galilei deduced rules of motion from experiment, Johannes Kepler from observation.
Newton and Leibniz introduced the mathematical tools for the flowering of the Scientific Epoch.

We have had four centuries of serious modelling of the physical world, of developing tools and insights to make the modelling effective and establishing the norms of
verification.

Yet somehow the new fields feel obliged to design their own version of the wheel.

2 The Equations

We introduce the equations of interest to us in this talk. Two of the equations are very familiar to scientists of all description and do not really require much in the way of introduction, but the other two come from disparate areas and may not be so familiar. We commence with them.

2.1 Economics: The Black-Scholes Equation

The practice of taking and exercising put and call options became common and acceptable in the Sixties of the last century. Earlier opinion had been that these were somewhat akin to gambling and were not appropriate activities for the Stock Exchange, perhaps in reaction to some of the excesses associated with the Great Crash of 1929.

In 1973 two seminal papers appeared which were devoted to the mathematical theory of option pricing. Black and Scholes developed a parabolic partial differential equation to describe the evolution in time of the value of what is known as an 'European option'. Black and Scholes obtained the equation

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}v^2x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \tag{2.1}$$

where $u(t,x)$ is the value of the option as a function of the time t and the stock price x. The parameters r and v^2 represents the risk-free interest rate and the variance of the rate of the return on the stock respectively. In the model as it stands these parameters are constants.

At about the same time as the appearance of the paper by Black and Scholes a mathematically somewhat more sophisticated paper was presented by Merton which was a substantial revision of some earlier work. The paper by Black and Scholes was received for publication towards the end of 1970 so that it would seem that both Merton and Black and Scholes were working closely in parallel.

Merton acknowledges the superiority of the Black-Scholes model in its provision of supplementary assumptions which enable a precision to be attached to the predictions of the model which were not available in his more general construct on mathematical principles and theorems.

In his Introduction Merton observes that, 'since options are specialised and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned.'

Already in their Conclusion Black and Scholes had observed that their results could be extended to many other situations and, in a sense, that virtually every financial instrument could be regarded in terms of an option.

Kwok observes that the revolution in derivative securities, from the early Seventies of the last century, has led to phenomenal growth in the field. The widespread growth of hedging as an attempt, not with complete success, to protect assets is indicative of this growth.

Underlying the development of the theory is the principle of riskless hedging. Black and Scholes made the following assumptions on the workings of the financial markets. They are

1. trading takes place continuously in time;

2. the riskless interest rate r is known and constant over time;

3. the asset pays no dividend;
4. there are no transactional costs in buying or selling the asset for the option and no taxes or other imperfections;
5. the assets are perfectly divisible;
6. there are no penalties attached to short selling and the full use of proceeds is permitted and
7. there are no riskless arbitrage opportunities.

This list of preconditions or assumptions is enough to make the moderately aware person wonder about what sort of a Dream World Black and Scholes indwelled. Nevertheless the assumptions provide a framework in which a mathematical model may be constructed.

The value of the mathematical model is to be measured in its predictions and their correlations with observations.

Black and Scholes reported the results of empirical tests of their formula on a large body of call-option data and the results of the tests indicated that there were systematic variations of reality from the prediction. Most of the deviation could be attributed to the large transaction costs of the options market. Of slighter importance was a difference for low-risk stocks compared with high-risk stocks. In fact it has shown that subsequent modifications to the Black-Scholes model have demonstrated that it is quite robust with respect to many of the assumptions above.

Equation (2.1) is an evolution equation with time as the evolution variable and the price of the underlying stock being, as it were, the spatial variable. Strictly speaking the Black-Scholes is a backwards evolution since it has a terminal condition, *videlicet*

$$u(T,x)=\max\{x_T-K,0\}, \tag{2.2}$$

where K is the exercise price of the option and x_T the value of the stock at the time, T, when the option is due to be exercised.

The Black-Scholes equation is linear. Thus it is reasonable to seek a transformation which renders it as the archetypal evolution equation in $1+1$ dimensions. Under the transformation

$$\tau=-v^2t,\quad \rho=\log x,\quad \phi(\rho,\tau)=\exp\left[\left(\frac{r}{v^2}-\frac{1}{2}\right)\rho\right]u(x,t) \tag{2.3}$$

(2.1) becomes

$$-\frac{\partial \phi}{\partial \tau} + \tfrac{1}{2}\frac{\partial^2 \phi}{\partial \rho^2} - w\phi = 0, \tag{2.4}$$

where $w = \dfrac{1}{2}\left(\dfrac{r}{v^2} + \dfrac{1}{2}\right)^2$.

The equation we are considering,

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}v^2x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0,$$

is the simplest form of the Black-Scholes equation.

In its original conception the equation dealt with the pricing of options, but it has become apparent that the equation has a much wider applicability.

2.2 Some Other Equations of Mathematical Finance

The source of most of my information is *Option Pricing, Interest Rates and Risk Management* (2001) Jouini Elyès, Cvitanić Jajusa & Musiela Murek edd (CUP, Cambridge) which is a compendium of papers to which specific reference is made below.

Goldys & Musiela (pp 314-335) present the backwards Kolmogorov equation, which is just a Black-Scholes equation (actually one should reverse the comment since Kolmogorov considerably preceded Black and Scholes), as

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0 \tag{2.5}$$

with the terminal condition $u(T,x) = (K-x)^+$.

On page 330 they introduce another form of this equation as

$$\frac{\partial u}{\partial t} + Lu - \delta(\phi)u = 0, \tag{2.6}$$

where $u = u(t,\phi)$ and now the terminal condition is $u(T,\phi) = F(\phi)$.

Heath, Platin & Schweizer present the equation

$$\frac{\partial v_{\hat{p}}}{\partial t} + \left(a - \frac{\rho b\mu}{y}\right)\frac{\partial v_{\hat{p}}}{\partial y} + \tfrac{1}{2}\left(x^2y^2\frac{\partial^2 v_{\hat{p}}}{\partial x^2} + b^2\frac{\partial^2 v_{\hat{p}}}{\partial y^2} + 2\rho xyb\frac{\partial^2 v_{\hat{p}}}{\partial x\partial y}\right) = 0 \tag{2.7}$$

on $(0,T)\times(0,\infty)\times R$ with terminal condition $v_{\hat{p}}(T,x,y)=h(x)$. This gives the pricing function for local risk-minimisation.

On page 516 they give an equation for mean-variance hedging as

$$\frac{\partial J}{\partial t}+a\frac{\partial J}{\partial y}+\tfrac{1}{2}b^2\frac{\partial^2 J}{\partial y^2}-\tfrac{1}{2}\left(\frac{\partial J}{\partial y}\right)^2+\left(\frac{\mu}{y}\right)^2=0 \tag{2.8}$$

with the terminal condition $J(T,y)=0$.

When (2.8) is solved, we then have the equation

$$\frac{\partial v_{\hat{p}}}{\partial t}+\left(a-b^2\frac{\partial J}{\partial y}\right)\frac{\partial v_{\hat{p}}}{\partial y}+\tfrac{1}{2}x^2y^2\frac{\partial^2 v_{\hat{p}}}{\partial x^2}+\tfrac{1}{2}\frac{\partial^2 v_{\hat{p}}}{\partial y^2}=0. \tag{2.9}$$

On page 523 they present a further equation, *videlicet*

$$\begin{aligned}&\frac{\partial u_{\hat{p}}}{\partial t}+\left[\left(\frac{\rho\kappa}{\Sigma}-\tfrac{1}{2}\right)y^2+\rho^2\Delta y-\frac{\rho\kappa\Delta}{\Sigma}\right]\frac{\partial u_{\hat{p}}}{\partial z}\\&+\left[\frac{4\kappa\beta-\Sigma^2}{8y}-\frac{\kappa y}{2}-\frac{\rho\Sigma\Delta}{2}\right]\frac{\partial u_{\hat{p}}}{\partial y}\\&+\tfrac{1}{2}y^2\left(1-\rho^2\right)\frac{\partial^2 u_{\hat{p}}}{\partial z^2}+\frac{\Sigma^2}{8}\frac{\partial^2 u_{\hat{p}}}{\partial y^2}=0\end{aligned} \tag{2.10}$$

in which one can start with ρ, κ, Σ, β and Δ constants.

Cvitanić informs us that the K-hedging price for $V(t,s)$ is the solution of the d-dimensional Black-Scholes equation

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}a_{ij}s_is_j\frac{\partial^2 V}{\partial s_i\partial s_j}+r\left(\sum_{i=1}^{d}s_i\frac{\partial V}{\partial s_i}-V\right)=0, \tag{2.11}$$

which is an equation of Hamilton-Jacobi-Bellman type with certain assumptions on the values of the parameters.

From an entirely different source we have another form of a Hamilton-Jacobian-Bellman equation as

$$J_t-\tfrac{1}{2}\frac{\mu^2}{\sigma^2}\frac{J_x^2}{J_{xx}}-\frac{\mu s v\rho}{\sigma}\frac{J_{xs}J_x}{J_{xx}}+\mu sJ_s-\tfrac{1}{2}s^2v^2\rho^2\frac{J_{xs}^2}{J_{xx}}+\tfrac{1}{2}s^2v^2J_{ss}=0, \tag{2.12}$$

where μ, σ, v and ρ are constant parameters.

We found an equation on a whiteboard in Greece. Unfortunately I cannot remember the particular application in Financial Mathematics which it describes, but there are certain aspects in terms of its symmetries which are worthy of report. The equation is

$$u_t + \tfrac{1}{2}\sigma^2 u_{xx} + (\alpha x + \beta u)\, u_x - (\gamma x + \delta u) = 0. \tag{2.13}$$

One must give these economists credit for being able to produce some of the uglier equations in existence!

2.3 Medicine: The Burgess equation

In the mathematical modelling of the spread of aggressive brain cancers such as glioblastoma multiforme Burgess *et al* make use of a spatiotemporal model described by the evolution equation

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + pn(r,t) - k_t n(r,t) \tag{2.14}$$

in which $n(r,t)$ is the concentration of tumour cells at location r at time t, D is the diffusion coefficient, *ie* a measure of the areal speed of the invading glioblastoma cells, p is the assumed constant rate of reproduction of the glioblastoma cells and k_t the killing rate of the same cells.

This last term has been used to investigate the effects of chemotherapy and surgery on survival. Note that the subscript refers to the abbreviation and not to a possible time dependence in the coefficient.

In this model the tumour is assumed to display spherical symmetry, the medium, through which it is expanding, to be isotropic and uniform and the replication rate of the glioblastoma cells to be Malthusian.

All of the assumptions above are open to criticism, but this spatiotemporal model is certainly superior to a simple temporal model in terms of its predictions. Consequently we accept the equation (2.14) with its assumptions for the purpose of the present analysis. However, one can investigate the effects of replacing the constant p with a simple spatially-dependent model. There is also the possibility to include time dependence, not so much in the replication rate as in the killing term k_t.

Under the changes of variables and parameters

$$\tau = 2Dt, \quad \phi(r,\tau) = rn(r,t), \quad w = \frac{k_t - p}{2D} \tag{2.15}$$

(2.14) becomes

$$-\frac{\partial \phi}{\partial \tau} = -\tfrac{1}{2}\frac{\partial^2 \phi}{\partial r^2} + w\phi. \tag{2.16}$$

Note that in general, *ie* when the proliferation and killing rates are functions of time and space, w is $w(r,t)$ without affecting (2.16).

We note the resemblance to (2.4)!

3 Extensions to other models

The form the Burgess equation is as it is found in the literature.

We consider other models for the Burgess equation.

We observed that the model developed by Burgess *et al* contained a Malthusian term consisting of two parts, artificially separated to enable comparisons of different effects, both of which were constant. We recall that the first term had the nature of a source since it contained the rate of replication of the glioblastoma cells and the second term the nature of a sink since it contained a killing term which could be due to intervention such as chemotherapy. We can consider the possibility of more complex models for this term. In the first instance we could examine the effect of a proliferation rate, p, which is not a constant but which depends upon the radial distance r. Another possibility is to examine the effect of a killing term, k_t, which is a function of time.

Obviously we could look at more involved model potentials in which source and sink terms are functions of both time and space. One could imagine, for example, that the very progression of the growth of the tumour could affect the proliferation rate in time as the host organism became debilitated by the parasitic presence of the tumour and was less able to supply the required nutrients. However, we suspect that in the case of a rampant cancer such as glioblastoma such fine detail is not of immediate relevance. A spatiotemporal variation of the killing term, k_t, is suggested by the use of some agent to kill the cancer cells. In the case of a biochemical agent the killing term would become dependent upon the concentration of the agent which itself would be the solution of a similar equation.

The assumption of a constant proliferation rate – the Malthusian model – does not take into account the effect of crowding or, alternatively, the amount space available for new cells. The constant proliferation rate is part of the assumption of growth in a uniform and isotropic medium. Without abandoning the incredibly – although not unreasonable – simplifying assumption of isotropy we can retreat from the assumption of uniformity by proposing that the proliferation rate be proportional to the radial distance.

In particular we assume a simple power law relationship of the form

$$p = cr^{\alpha}, \quad \alpha > 0, \tag{3.1}$$

which simply suggests that there is more room for cell division the further one is from the centre of the tumour. Possible values for α which are mathematically amenable are $\alpha = 1$ and $\alpha = 2$. In the latter one assumes that the space available is proportional to the surface area at radius r whereas in the former one assumes that the effect of increased surface area is somehow diminished. For the moment we keep the general power, α, to obviate the necessity to write multiple versions of almost the same equation.

The equation corresponding to (2.16) is now

$$-\frac{\partial \phi}{\partial \tau} = -\tfrac{1}{2}\frac{\partial^2 \phi}{\partial r^2} + w\phi, \tag{3.2}$$

where now

$$w = k_t - cr^{\alpha}. \tag{3.3}$$

The classical Lagrangian corresponding to (3.2) with (3.3) is

$$L = \tfrac{1}{2}\dot{r}^2 - \left(k_t - cr^{\alpha}\right), \tag{3.4}$$

which is not a very attractive system from the point of view of Classical Mechanics since the potential is repulsive.

Fortunately Noether's Theorem is not concerned with the attractiveness of a potential but merely its functional form[1]. In the cases of both $\alpha = 1$ and $\alpha = 2$ the Lagrangian (3.4) is still one with the maximal number of Noether point symmetries, *ie* five. When $\alpha = 1$, we have motion in a uniform force field, as in the motion in the constant gravitational field near the surface of the Earth. The classical equations of the motion were obtained experimentally by Galileo and reported by him in 1604[2]. When $\alpha = 2$, we

[1] There are times when we must be reminded of the distinction between Mathematics and Physics. The latter discipline is habitually rooted in reality.

[2] This experimental derivation is, quite properly, cited as one of the important evidences of the development of the New Science based upon the discipline of experimental verification rather than the attractions of philosophical niceties. Galileo's experiment persists to this day. So do the constant acceleration formulæ which summarise Galileo's experimental results. It is unfortunate that students take to these formulæ with a passion so persistent that for too many of them it precludes the possibility of any other solution to Newtonian equations of motion.

have a linear repulsor, the not so harmonic relation to the paradigmatic simple harmonic oscillator.

We can move from the Noether point symmetries of the Lagrangian (3.4) to the Lie point symmetries of the modified Burgess equation by the standard relationship between the classical Noether symmetries of the Lagrangian and the Lie symmetries of the Schrödinger equation and the further transformations to the Burgess equation with the new model proliferation rate.

3.1 Chemistry: The Schrödinger equation

In $1+1$ dimensions the Schrödinger equation takes the form

$$i\frac{\partial \phi}{\partial t} = -\tfrac{1}{2}\frac{\partial^2 \phi}{\partial x^2} + V(x,t)\phi, \tag{3.5}$$

in which we have allowed for a general variable dependence in the potential, corresponding to the classical Hamiltonian and Lagrangian

$$H = \tfrac{1}{2}p^2 + V(x,t) \tag{3.6}$$

$$L = \tfrac{1}{2}\dot{x}^2 - V(x,t), \quad p = \dot{x}. \tag{3.7}$$

3.2 Physics: The heat equation

The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t,x)u \tag{3.8}$$

is the $1+1$ heat equation with sources and/or sinks depending upon both location and time. The equation could also have a term in $\partial u/\partial x$.

The mathematical properties of the equations we have listed are the same since the equations are essentially the same.

Naturally one needs to keep an eye on the boundary/initial conditions, but we leave that for another story.

4 Symmetry

The idea of a symmetry in the simplest sense relates to regularity in the structure of an object. If an object is regular then it has aspects that persist under types of transformation, for example in Geometry the equilateral triangle and the circle are invariant under rotations. In Physics the laws of nature are invariant under time and space translation. In Mathematics some of these objects of invariance are differential equations.

Lie group theory is the study of symmetry properties of these abstract structures. The original formulation some 130 years ago was by Sophus Lie, a Norwegian, through the study of infinitesimal transformations which lead to continuous transformation groups and symmetry groups. The discovery in 1918 by Noether of the relationship between continuous symmetries and conservation laws has entrenched the ideas in the theory and application of differential equations.

Symmetries provide information on regular objects under transformation. If that object be a differential equation, the symmetry group can be used to find invariants, transformations, reductions and solutions of the differential equation.

5 The Classical Connection

5.1 Conservation Laws

Hamiltonians/Lagrangians/Euler-Lagrange equations can possess a variety of conservation laws.

Some typical examples are:

- $$\frac{\partial H}{\partial t} = 0 \quad \Leftrightarrow \quad H = H(q,p).$$

  The energy ($= H$) is conserved.

  $\partial H/\partial t = 0$ means that there is no change in H as time changes. We say that H is invariant under time translation, that it possesses the symmetry of time translation, ∂_t ($:= \partial/\partial t$).

- $$H = H(r, \mathbf{p}), \quad ie \quad \frac{\partial H}{\partial \theta} = 0,\ \frac{\partial H}{\partial \phi} = 0.$$

  There is spherical symmetry and the conservation of angular momentum which is invariant under the transformations associated with the rotation group.

The algebraic representation of the group in terms of differential operators is

$$\begin{aligned}\Gamma_1 &= \partial_\phi \\ \Gamma_{2\pm} &= \mathrm{e}^{\pm i\phi}\left(\partial_\theta \pm i\cot\theta\partial_\phi\right).\end{aligned}$$

5.2 Emmy Noether

The direct relationship between the existence of a conserved quantity – first integral/invariant – and invariance under some type of transformation – e.g. time translation or rotation – was formalised by Emmy Noether in 1918.

We need some formalities:

Infinitesimal transformation

$$\begin{array}{l}\bar{t} = t + \epsilon\tau \\ \bar{x} = x + \epsilon\eta\end{array} \quad \Leftrightarrow \quad \begin{array}{l}\bar{t} = (1 + \epsilon\Gamma)\, t \\ \bar{x} = (1 + \epsilon\Gamma)\, x,\end{array} \tag{5.1}$$

where the infinitesimal transformation is generated by the differential operator

$$\Gamma = \tau\partial_t + \eta\partial_x. \tag{5.2}$$

In particular

$$\Gamma t = \tau \quad \text{and} \quad \Gamma x = \eta. \tag{5.3}$$

When Γ is associated with an invariant such as time translation and the consequent conservation of energy, it is called a symmetry.

5.3 Noether's Theorem

The Action Integral

$$A = \int_{t_0}^{t_1} L(t, x, \dot{x})\mathrm{d}t \tag{5.4}$$

possesses a Noether symmetry of the form

$$\Gamma = \tau\partial_t + \eta\partial_x \tag{5.5}$$

if it is invariant under the infinitesimal transformation, $\bar{t} = t+\epsilon\tau,\ \bar{x} = x+\epsilon\eta$, generated by Γ.

If A possesses a Noether symmetry, then there exists a conserved quantity

$$I = f - \left[\tau L + (\eta - \dot{x}\tau)\frac{\partial L}{\partial \dot{x}}\right] \tag{5.6}$$

$$= f + \tau H - \eta p. \tag{5.7}$$

5.4 Comparison of Classical Mechanics and Evolution Equations

The results regarding the number of Lie point symmetries of a linear evolution equation – indeed linearisable – remind one of the possible cases for the number of the Noetherian symmetries of a classical Lagrangian of the form

$$L = \tfrac{1}{2}\dot{x} - V(t,x). \tag{5.8}$$

The possibilities are

V	*no sym*	*algebra*
$V(t,x)$	0	$-$
$V(x)$	1	A_1
$\omega^2 x^2 + \dfrac{h^2}{x^2}$	3	$sl(2,R)$
$\omega^2 x^2$	5	$sl(2,R) \oplus_s 2A_1$.

This relates to the Lie point symmetries of the corresponding time-dependent Schrödinger Equation

$$2i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - V(t,x)u = 0 \tag{5.9}$$

for which we have

V	*no sym*	*algebra*
$V(t,x)$	$0+1+\infty$	$A_1 \oplus_s \infty A_1$
$V(x)$	$1+1+\infty$	$2A_1 \oplus_s \infty A_1$
$\omega^2 x^2 + \dfrac{h^2}{x^2}$	$3+1+\infty$	$\{A_1 \oplus sl(2,R)\} \oplus_s \infty A_1$
$\omega^2 x^2$	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$,

where W is the Weyl algebra with

$$[\Sigma_1, \Sigma_2]_{LB} = 0,\ [\Sigma_1, \Sigma_3]_{LB} = 0,\ [\Sigma_2, \Sigma_3]_{LB} = \Sigma_1.$$

Naturally the time-dependent Schrödinger Equation is transformed to the heat equation by a point transformation. Subsequent transformation, as we have seen, brings us to the various forms of the Black-Scholes equation, not necessarily with $5 + 1 + \infty$ Lie point symmetries which may well have been the attraction of the symmetry analysis of the Black-Scholes equation in the first place.

6 Last Comments

A point which we have not addressed has been the matter of boundary and initial/terminal conditions.

One method, which is found in the literature, is to include the boundaries as part of the problem in the determination of symmetries.

Another approach is to attempt to use the infinite sets of solutions which we can obtain for some problems – those with a reasonable amount of symmetry – and see if they can be used as a basis set.

A third approach is to determine all the symmetries and then to see which combinations of them are consistent with the boundary and initial/terminal conditions.

All three methods have their own relevance in different contexts.

In this context we have so far mentioned only the (1+1)-dimensional equations of Financial Mathematics, Medicine, Physics and Chemistry. However, there is no necessity to restrict the considerations to such problems. Indeed it would be foolish so to do.

One consequence of the relationship of the partial differential equations considered here to Lagrangians of Rational Mechanics has been the existence of symmetries of the partial differential equations which correspond to Noether symmetries of the Action Integral. As a corollary one could imagine quite easily that the absence of symmetries in partial differential equations of the type considered here is matched by an absence of symmetry in the corresponding classical mechanical problem. An absence of symmetry in multidimensional mechanical systems is an indication of the possibility of the presence of chaos.

Consequently we must look at the equivalent in Classical Mechanics. If we are in two spatial dimensions, there is the possibility of chaos if the potential departs even only modestly from the quadratic. Classical chaos has its counterpart in the so-called Quantum Chaos which has been the object of so much investigation and discussion over the last few decades. Given the connection already indicated several times above between the Schrödinger equation and the equations considered here we must ask whether they can display an equivalent of quantum chaos.

Is it then possible that we can exhibit economical and biological models which display something akin to chaos in their solutions?

If this be the case, one immediate question is whether we would have Chaotic Economics or Economical Chaos?