Complete Symmetry Groups

PGL Leach

School of Mathematical and Statistical Sciences
Howard College, University of KwaZulu-Natal
Durban 4041, Republic of South Africa

in conjunction with

K Andriopoulos and MC Nucci

Department of Physics

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1 The idea

The term 'Complete symmetry group' has its history.

1994 J Krause: 'On the complete symmetry group of the classical Kepler system': a complete symmetry group of a differential equation is the group associated with the set of symmetries, be they point, contact, generalised or nonlocal, required to specify the equation or system completely. A realisation of a 'complete symmetry group' must be endowed with the following two properties:

(a) the group acts freely and transitively on the manifold of all allowed motions of the system (*ie* the manifold of solutions is an homogeneous space of the group) and

(b) the given equations of motion are the only ordinary differential equations that remain invariant under the specified action of the group (*ie* the group be specific to the system, no other system admit it).

That means that every mechanical system can be completely characterised by the symmetry laws it obeys, *ie* , the group of symmetries characterising a given system would be 'complete'.

2 A simple example: the Ermakov-Pinney equation.

$$\ddot{x} + x = \frac{k}{x^3}. \tag{2.1}$$

Three point symmetries

$$\begin{aligned} G_1 &= \partial_x \\ G_{2\pm} &= \mathrm{e}^{\pm 2it}\partial_t \pm ix\mathrm{e}^{2it}\partial_x \end{aligned}$$

Consider

$$\ddot{x} = f(t, x, \dot{x}).$$

The action of the symmetry $\Gamma_1 = \partial_t$ is

$$f = g(x, \dot{x}). \tag{2.2}$$

$$\Gamma_{2+}^{[2]} = \mathrm{e}^{2it}\partial_t + ix\mathrm{e}^{2it}\partial_x + (-i\dot{x}\mathrm{e}^{2it} - 2x\mathrm{e}^{2it})\partial_{\dot{x}} + (-3i\ddot{x}\mathrm{e}^{2it} - 4ix\mathrm{e}^{2it})\partial_{\ddot{x}}$$

the action of which on (2.2) gives

$$\begin{aligned} -3i\ddot{x}\mathrm{e}^{2it} - 4ix\mathrm{e}^{2it} &= ix\mathrm{e}^{2it}\frac{\partial g}{\partial x} + (-i\dot{x}\mathrm{e}^{2it} - 2x\mathrm{e}^{2it})\frac{\partial g}{\partial \dot{x}} \\ 3g + 4x &= -x\frac{\partial g}{\partial x} + (\dot{x} - 2ix)\frac{\partial g}{\partial \dot{x}}. \end{aligned}$$

This has the associated Lagrange's system

$$\frac{\mathrm{d}x}{-x} = \frac{\mathrm{d}\dot{x}}{\dot{x} - 2ix} = \frac{\mathrm{d}g}{3g + 4x},$$

from which we determine the two characteristics

$$u = x\dot{x} - ix^2 \quad \text{and} \quad v = x^3 g + x^4$$

so that

$$g = -x + \frac{1}{x^3} h(x\dot{x} - ix^2), \tag{2.3}$$

where h is an arbitrary function of its argument.

$$\Gamma_{2-}^{[2]} = \mathrm{e}^{-2it}\partial_t - ix\mathrm{e}^{-2it}\partial_x + (i\dot{x}\mathrm{e}^{-2it} - 2x\mathrm{e}^{-2it})\partial_{\dot{x}} + (3i\ddot{x}\mathrm{e}^{-2it} + 4ix\mathrm{e}^{-2it})\partial_{\ddot{x}},$$

the action of which on (2.3) gives

$$3i\ddot{x} + 4ix =$$

$$ix + 3ix\frac{1}{x^4}h + \frac{1}{x^3}\frac{\partial h}{\partial u}\left(-\dot{x}ix + x\left(ix - 2x\right) - 2x^2\right)$$

$$3i\left(-x + \frac{h}{x^3}\right) + 3ix = 3i\frac{1}{x^3}h + \frac{1}{x^3}\frac{\partial h}{\partial u}\left(-4x^2\right)$$

$$\frac{\partial h}{\partial u} = 0$$

$$h = k.$$

The Ermakov-Pinney equation is completely specified by Γ_1 and $\Gamma_{2\pm}$.

3 A first integral

$I_1 = y'$ of $y'' = 0$ has the symmetries

$$G_1 = \partial_x \quad G_2 = \partial_y \quad G_3 = x\partial_x + y\partial_y.$$

Let

$$I = f(x, y, y') \tag{3.1}$$

be a general function of the first order. The action of G_1 gives $f = g(y, y')$ and the action of G_2 gives $g = h(y')$. If we now apply G_3, we obtain $G_3^{[1]} h(y') = 0$. If we took firstly the action of G_3 on (3.1), we would obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 0\frac{\partial f}{\partial y'} = 0$$

with the two obvious characteristics

$$u = \frac{y}{x}, \quad v = y'$$

so that

$$f(x, y, y') = g\left(\frac{y}{x}, y'\right).$$

The action, say of G_1 would give

$$\frac{\partial g}{\partial u}\left(-\frac{y}{x^2}\right) = 0 \quad \Rightarrow \quad g = h(y').$$

Any two of the three symmetries are sufficient to specify it up to an arbitrary function of the integral. Note that $[G_1, G_2] = 0$ and $[G_1, G_3] = G_1$ with the algebra $2A_1$ (Type I) and A_2 (Type III) respectively.

4 Complete symmetry groups of ordinary differential equations and their integrals: some basic considerations

4.1 Second-order ordinary differential equations

For the equation

$$y'' = 0, \tag{4.1}$$

$$\begin{array}{ll} G_1 = \partial_y & G_5 = x\partial_x + \frac{1}{2}y\partial_y \\ G_2 = x\partial_y & G_6 = x^2\partial_x + xy\partial_y \\ G_3 = y\partial_y & G_7 = y\partial_x \\ G_4 = \partial_x & G_8 = xy\partial_x + y^2\partial_y. \end{array} \tag{4.2}$$

Consider

$$y'' = f(x, y, y')$$

and impose the symmetries of (4.2) in turn until we recover (4.1). Under G_1 the function f becomes $f(x, y')$. The y' is removed from f when G_2 is applied. Finally the effect of G_3 makes f zero. The Lie Brackets are

$$[G_1, G_2] = 0, \quad [G_1, G_3] = G_1, \quad [G_2, G_3] = G_2 \tag{4.3}$$

which is the algebra $D \oplus_s T_2$.

G_3, G_7 and G_8 also suffice and the algebra is $A_{3,3}$.

The $sl(2, R)$ symmetries reduce $y'' = f(x, y, y')$ to

$$y'' = \frac{K}{y^3}. \tag{4.4}$$

4.2 A nonlocal example

$$y'' + kyy' + y^3 = 0 \tag{4.5}$$

$$G_1 = \partial_x \quad \text{and} \quad G_2 = -x\partial_x + y\partial_y. \tag{4.6}$$

The two symmetries

$$y'' = f(x, y, y')$$

to the form

$$y'' = y^3 f\left(\frac{y'}{y^2}\right). \tag{4.7}$$

To find the third symmetry needed assume that

$$G_3 = \xi\partial_x \tag{4.8}$$

without any restriction on the functional dependence of ξ. The condition that G_3 be a symmetry of (4.5) is

$$2y''\xi' + y'\xi'' + kyy'\xi' = 0$$

which is a linear first order equation in the variable ξ' and

$$\xi = \int \frac{\exp\left[-\int ky\mathrm{d}x\right]}{y'^2}\mathrm{d}x. \tag{4.9}$$

Do the calculation.

4.3 The integrals of second-order ordinary differential equations

An Emden-Fowler equation

$$y'' + 2y^3 = 0 \tag{4.10}$$

has the obvious integral

$$I = y'^2 + y^4. \tag{4.11}$$

and the obvious point symmetry ∂_x. Although (4.10) has a second point symmetry, this is not a symmetry of the integral. We assume a symmetry of the form

$$G = \eta\partial_y \tag{4.12}$$

and apply its first extension to (4.11) to obtain

$$\frac{\eta'}{\eta} = -2\frac{y^3}{y'}$$

which has the solution

$$\eta = \exp\left[-2\int\frac{y^3}{y'}\mathrm{d}x\right]. \tag{4.13}$$

If we consider a general function $f(x, y, y')$, the action of the first symmetry, ∂_x, gives $f = f(y, y')$. The action of the first extension of the second symmetry, (4.12) with η as given in (4.13), on this function gives

$$\exp\left[-2\int\frac{y^3}{y'}\mathrm{d}x\right]\left(\frac{\partial f}{\partial y} - 2\frac{y^3}{y'}\frac{\partial f}{\partial y'}\right) = 0.$$

The equation for the characteristic is simply

$$0 = y'\mathrm{d}y' + 2y^3\mathrm{d}y$$

and the integral follows immediately.

In (4.13) we have written the coefficient function, η, in nonlocal form. Equally we could have invoked the differential equation to write

$$\eta = \exp\left[\int \frac{y''}{y'}\mathrm{d}x\right] = y',$$

so that the nonlocal symmetry is equivalent to a generalised symmetry. When we apply the first extension of the generalised symmetry to the integral, it is necessary to take the differential equation into account. We note that there exists a third symmetry of the differential equation of the form

$$G = \left(\int \frac{\mathrm{d}x}{y'^2}\right)\partial_x$$

which is not a symmetry of the integral. Under the conventional route of reduction of the order of (4.10) using the symmetry ∂_x this symmetry becomes local and so is a hidden symmetry of Type II.

We observe that it appears as if the complete symmetry group of a first integral of a second order differential equation is represented by an algebra with two elements. The group is not necessarily unique.

5 A Two-dimensional System of Beloved Equations

The one-dimensional system

$$\ddot{x} + 3x\dot{x} + x^3 = 0 \tag{5.1}$$

is linearisable by means of the point transformation

$$T = t - \frac{1}{x} \quad X = \tfrac{1}{2}t^2 - \frac{t}{x}. \tag{5.2}$$

For a higher-dimensional system, say

$$\begin{aligned} \ddot{x} + 3x\dot{x} + x^3 &= 0 \\ \ddot{y} + 3y\dot{y} + y^3 &= 0, \end{aligned} \tag{5.3}$$

this approach is not possible.

We linearise via the Riccati transformation

$$x = \frac{\dot{u}}{u} \quad y = \frac{\dot{v}}{v} \tag{5.4}$$

so that the system (5.3) becomes

$$\dddot{u} = 0 \quad \dddot{v} = 0. \tag{5.5}$$

Lie point symmetries of (5.3) are

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= t\partial_t - x\partial_x - y\partial_y \\ \Gamma_3 &= \tfrac{1}{2}t^2\partial_t + (1 - tx)\partial_x + (1 - ty)\partial_y \end{aligned} \tag{5.6}$$

and they possess the algebra $sl(2, R)$.

The system (5.5) has no intrinsically contact symmetries. The thirteen Lie point symmetries are

Symmetry	Algebra
$\Lambda_1 = \partial_u$	
$\Lambda_2 = t\partial_u$	$3A_1$
$\Lambda_3 = \frac{1}{2}t\partial_u$	
$\Lambda_4 = \partial_v$	
$\Lambda_5 = t\partial_v$	$3A_1$
$\Lambda_6 = \frac{1}{2}t\partial_v$	
$\Lambda_7 = u\partial_u - v\partial_v$	
$\Lambda_8 = u\partial_v - v\partial_u$	$so(2,1) \otimes A_1$
$\Lambda_9 = u\partial_v + v\partial_u$	
$\Lambda_{10} = u\partial_u + v\partial_v$	
$\Lambda_{11} = \partial_t$	
$\Lambda_{12} = t\partial_t + u\partial_u + v\partial_v$	$sl(2,R)$
$\Lambda_{13} = \frac{1}{2}t^2\partial_t + t(u\partial_u + v\partial_v)$	

(5.7)

with algebra $\{sl(2,R)\otimes\{so(2,1)\otimes A_1\}\}\otimes_s\{3A_1\otimes 3A_1\}$.

We note that the $sl(2,R)$ subalgebra of Λ_{11}, Λ_{12} and Λ_{13} is the one preserved in the decrease of order using the nonlocal transformation (5.4).

What happens to the other ten? A symmetry of (5.3), $\Sigma = \tau\partial_t + \xi\partial_x + \eta\partial_y$, is related to a symmetry of (5.5), $\Lambda = T\partial_t + \Xi\partial_u + H\partial_v$, according to

$$\begin{aligned}
\tau &= T \\
\xi &= \left(\frac{\dot{\Xi}}{u}\right) - \dot{T}x \\
\eta &= \left(\frac{\dot{H}}{v}\right) - \dot{T}y
\end{aligned} \tag{5.8}$$

between the coefficient functions.

For the "solution" symmetries we have

$$\begin{aligned}
\Lambda_1 &\Longrightarrow \Sigma_1 = -x\exp[-\textstyle\int x\mathrm{d}t] \\
\Lambda_2 &\Longrightarrow \Sigma_2 = (1-tx)\exp[-\textstyle\int x\mathrm{d}t] \\
\Lambda_3 &\Longrightarrow \Sigma_3 = \left(t-\tfrac{1}{2}t^2x\right)\exp[-\textstyle\int x\mathrm{d}t] \\
\Lambda_4 &\Longrightarrow \Sigma_4 = -y\exp[-\textstyle\int y\mathrm{d}t] \\
\Lambda_5 &\Longrightarrow \Sigma_5 = (1-ty)\exp[-\textstyle\int y\mathrm{d}t] \\
\Lambda_6 &\Longrightarrow \Sigma_6 = \left(t-\tfrac{1}{2}t^2y\right)\exp[-\textstyle\int y\mathrm{d}t].
\end{aligned} \tag{5.9}$$

In the case of the four-dimensional subalgebra the Λ_7 and Λ_{10} elements are used for the nonlocal transformation (5.4). For the other two we have

$$\begin{aligned}
\Lambda_8 &\Longrightarrow \Sigma_8 = (x-y)\left[\exp[-\textstyle\int(x-y)\mathrm{d}t]\partial_x + \exp[\textstyle\int(x-y)\mathrm{d}t]\partial_y\right] \\
\Lambda_9 &\Longrightarrow \Sigma_9 = (x-y)\left[\exp[-\textstyle\int(x-y)\mathrm{d}t]\partial_x - \exp[\textstyle\int(x-y)\mathrm{d}t]\partial_y\right].
\end{aligned}$$

The remaining three symmetries transform as

$$\begin{aligned}
\Lambda_{11} &\Longrightarrow \Sigma_{11} = \partial_t \\
\Lambda_{12} &\Longrightarrow \Sigma_{12} = t\partial_t - x\partial_x - y\partial_y \qquad (5.10)\\
\Lambda_{13} &\Longrightarrow \Sigma_{13} = \tfrac{1}{2}t^2\partial_t - t\left(x\partial_x + y\partial_y\right).
\end{aligned}$$

In the representation of the third-order system, (5.5), the Lie Bracket of Λ_8 and Λ_9 is

$$[\Lambda_8, \Lambda_9]_{LB} = 2\Lambda_7.$$

What happens in the case of the representation for the second-order system, (5.3), since there is no expression for Λ_7?

5.1 Doing the wrong thing

Amongst some pundits it is an article of faith that one does not use a nonnormal subgroup for reduction of order.

We reduce order of the first member of (5.5) using Σ_1. The associated Lagrange's system is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}\dot{x}}{\dot{x} - x^2}$$

after one has removed the common factor of $\exp[-\int x\mathrm{d}t]$. The characteristics are t and

$$w = \frac{\dot{x}}{x} + x. \tag{5.11}$$

The reduced equation is

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -w^2$$

which is easily solved to give

$$w = \frac{1}{K+t}.$$

When we combine this with (5.11) to obtain

$$\dot{\left(\frac{1}{x}\right)} + \frac{1}{K+t}\left(\frac{1}{x}\right) = 1$$

which has the solution

$$x = \frac{K+t}{C + \frac{1}{2}(K+t)^2}. \tag{5.12}$$

5.2 Complete symmetry group

$$\begin{aligned} \ddot{x} + 3x\dot{x} + x^3 &= 0 \\ \ddot{y} + 3y\dot{y} + y^3 &= 0. \end{aligned} \tag{5.13}$$

We expect five symmetries to specify it completely. From previous experience with $sl(2, R)$ one would not expect that it would give the result – at least in part – for it is not a very efficient algebra in this respect. We must go nonlocal. For an initial selection we take the "solution" symmetries

$$\begin{aligned} \Sigma_1 &= -x\exp[-\textstyle\int x\mathrm{d}t] & \Sigma_4 &= -y\exp[-\textstyle\int y\mathrm{d}t] \\ \Sigma_2 &= (1-tx)\exp[-\textstyle\int x\mathrm{d}t] & \Sigma_5 &= (1-ty)\exp[-\textstyle\int y\mathrm{d}t]. \end{aligned} \tag{5.14}$$

We consider the system

$$\ddot{x} = f(t, x, y, \dot{x}, \dot{y}) \tag{5.15}$$

$$\ddot{y} = g(t, x, y, \dot{x}, \dot{y}). \tag{5.16}$$

We commence with Σ_1 and its effect on (5.15). The second extension is

$$\Sigma_1^{[2]} = \exp[-\textstyle\int x\mathrm{d}t]\left\{x\partial_x + \left(\dot{x} - x^2\right)\partial_{\dot{x}} + \left(\ddot{x} - 3x\dot{x} + x^3\right)\partial_{\ddot{x}}\right\} \tag{5.17}$$

and its action on (5.15) leads to the associated Lagrange's system

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}\dot{x}}{\dot{x} - x^2} = \frac{\mathrm{d}f}{f - 3x\dot{x} + x^3} \tag{5.18}$$

for which the characteristics are t, y, $\dot{y}$, $w = x + \dot{x}/x$ and $v = 3wx - 2x^3 + f/x$. Equation (5.15) is now

$$\ddot{x} = xf_1(t, y, \dot{y}, w) - 3x\dot{x} - x^3. \tag{5.19}$$

The second extension of Σ_2 is

$$\begin{aligned}\Sigma_2^{[2]} &= \exp[-\textstyle\int x\mathrm{d}t]\left\{(1 - tx)\partial_x + \left[-2x - t\left(\dot{x} - x^2\right)\right]\partial_{\dot{x}}\right.\\ &\quad \left. + \left[-3\left(\dot{x} - x^2\right) - t\left(\ddot{x} - 3x\dot{x} + x^3\right)\right]\partial_{\ddot{x}}\right\}\end{aligned} \tag{5.20}$$

in which we see that part of the expression is simply $-t\Sigma_1^{[2]}$ so that we can use

$$\Sigma_{2(\mathrm{eff})}^{[2]} = \exp[-\textstyle\int x\mathrm{d}t]\left\{\partial_x - 2x\partial_{\dot{x}} - 3\left(\dot{x} - x^2\right)\partial_{\ddot{x}}\right\}. \tag{5.21}$$

The action of (5.21) on (5.19) leads to the equation

$$w\frac{\partial f_1}{\partial w} - f_1 = 0$$

with the solution $f_1 = K(t, y, \dot{y})w$ so that (5.19) becomes

$$\ddot{x} + 3x\dot{x} + x^3 = \left(\dot{x} + x^2\right)K(t, y, \dot{y}). \tag{5.22}$$

The second extension of Σ_4 is

$$\Sigma_4^{[2]} = \exp[-\textstyle\int y\mathrm{d}t]\left\{y\partial_y + \left(\dot{y} - y^2\right)\partial_{\dot{y}} + \left(\ddot{y} - 3y\dot{y} + y^3\right)\partial_{\ddot{y}}\right\} \tag{5.23}$$

and its action on (5.22) gives

$$0=\left(\dot{x}+x^2\right)\left[y\frac{\partial K}{\partial y}+\left(\dot{y}-y^2\right)\frac{\partial K}{\partial \dot{y}}\right]$$

so that $K=K_1(t,r)$, where $r=y+\dot{y}/y$, and (5.22) is

$$\ddot{x}+3x\dot{x}+x^3=\left(\dot{x}+x^2\right)K_1(t,r). \tag{5.24}$$

As in the case of Σ_2 we may use the second extension of Σ_4 to give us

$$\Sigma_{5(\mathrm{eff})}^{[2]}=\exp[-\textstyle\int y\mathrm{d}t]\left\{\partial_y-2y\partial_{\dot{y}}-3\left(\dot{y}-y^2\right)\partial_{\ddot{y}}\right\}. \tag{5.25}$$

The action of this on (5.24) gives

$$r\frac{\partial K}{\partial r}=0 \quad\Longrightarrow\quad K_1=K_2(t)$$

so that (5.24) is now

$$\ddot{x}+3x\dot{x}+x^3=\left(\dot{x}+x^2\right)K_2(t). \tag{5.26}$$

In a similar process one reduces (5.16) to

$$\ddot{y}+3y\dot{y}+y^3=\left(\dot{y}+y^2\right)L_2(t). \tag{5.27}$$

To finish the job we combine Σ_3 and Σ_6. Thus we have

$$\begin{aligned}\tilde{\Sigma}_{(\mathrm{eff})}^{[2]} &= \Sigma_{3(\mathrm{eff})}^{[2]}+\Sigma_{3(\mathrm{eff})}^{[2]}\\ &= \exp[-\textstyle\int x\mathrm{d}t]\left(\partial_{\dot{x}}-3x\partial_{\ddot{x}}\right)+\exp[-\textstyle\int y\mathrm{d}t]\left(\partial_{\dot{y}}-3y\partial_{\ddot{y}}\right)\end{aligned} \tag{5.28}$$

and the action of this on (5.26) gives

$$-3x + 3x = K_2 \quad \Longrightarrow \quad K_2 = 0$$

and similarly on (5.27) we obtain $L_2 = 0$.

Thus we have found the five symmetries necessary to specify completely the system (5.13).

Can we combine the other symmetries as in the case of Σ_3 and Σ_6? This is not the case. It is not possible to combine any symmetries if both variables are present in any one equation. The effectiveness of the method requires the removal of the nonlocal term and, when both variables are present, this is not possible. For the final act each equation had been reduced to one dependent variable only and so the combination of the two nonlocal terms in the one symmetry was irrelevant.

Obviously one could use any linear combination of Σ_3 and Σ_6.

There is an amusing observation. One representation of the complete symmetry group of the system (5.13), *ie* the one which we have just obtained, is completely in terms of nonlocal symmetries.

This is another example of a superintegrable system which possesses fewer point symmetries than it theoretically should have for complete reduction.

5.3 Algebra of the complete symmetry group

We need to introduce two variables to replace the nonlocality and to introduce two extra terms into the symmetry. We take them to be

$$I = \int x\mathrm{d}t \quad \text{and} \quad J = \int y\mathrm{d}t. \tag{5.29}$$

Then a symmetry in the new variables, say

$$\Omega = \Xi\partial_I + H\partial_J + \tau\partial_t, \tag{5.30}$$

when once extended gives

$$\Omega^{[1]} = \Omega + \left(\dot{\Xi} - \dot{\tau}x\right)\partial_x + \left(\dot{H} - \dot{\tau}y\right)\partial_y. \tag{5.31}$$

The symmetry coincides with a standard form

$$\omega = \tau\partial_t + \xi\partial_x + \eta\partial_y \tag{5.32}$$

on the identifications

$$\xi = \dot{\Xi} - \dot{\tau}x \quad \text{and} \quad \eta = \dot{H} - \dot{\tau}y. \tag{5.33}$$

It is not a difficult task to find the new representation in the case of the solution symmetries since $\tau = 0$ for all of them. We have

$$\begin{array}{ll}
\Omega_1 = \exp[-I]\left(\partial_I - x\partial_x\right) & \Omega_4 = \exp[-J]\left(\partial_J - y\partial_y\right) \\
\Omega_2 = \exp[-I]\left(t\partial_I + (1 - tx)\partial_x\right) & \Omega_5 = \exp[-J]\left(t\partial_J + (1 - ty)\right. \\
\Omega_3 = \exp[-I]\left(\frac{1}{2}t^2\partial_I + (t - \frac{1}{2}t^2x)\partial_x\right) & \Omega_6 = \exp[-J]\left(\frac{1}{2}t^2\partial_J + (t - \frac{1}{2}\right.
\end{array} \tag{5.34}$$

The Lie Brackets of these symmetries are all zero. The Complete Symmetry Group is abelian.

5.4 The missing Lie Bracket

We are now in a position to calculate the Lie Bracket of Σ_8 and Σ_9. We use the above to write them in terms of the 'nonlocal' variables I and J. We have

$$\begin{aligned} \Sigma_8 &= \exp[-(I-J)]\partial_I + \exp[(I-J)]\partial_J \\ \Sigma_9 &= \exp[-(I-J)]\partial_I + \exp[(I-J)]\partial_J, \end{aligned} \quad (5.35)$$

where we have at last realised that it is not necessary to include the x and y parts since they are the first extension and we recall that the Lie Bracket of the extended operators is the extension of the Lie Bracket of the operators. We can now calculate the Lie Bracket in the normal way as

$$[\Sigma_8, \Sigma_9]_{LB} = 2\left(\partial_I - \partial_J\right). \quad (5.36)$$

The Lie Bracket is not zero, but as far as the present local variables are concerned it is.